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A Rigidity Theorem for Finite Group Actions on Enveloping Algebras of Semisimple Lie Algebras

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Let $\mathfrak{g}$ denote a semisimple Lie algebra over an algebraically closed field k of characteristic zero and G, a finite group of k-automorphisms of the enveloping algebra U of $\mathfrak{g}$. In this paper, it is proved that, if the subalgebra U^G is k-isomorphic to an enveloping algebra, then G is trivial. A similar result for Weyl algebras over k is also obtained.

INTRODUCTION

Let k denote an algebraically closed field of characteristic zero. A well-known theorem of Shephard Todd and Chevalley asserts that, if G is a finite group of automorphisms of a finite dimensional vector space V over k, then the algebra of invariants of G acting in the symmetric algebra of V is a polynomial algebra over k if and only if G is generated by pseudoreflections. In searching for eventual noncommutative analogues of this result, the following rigidity property is observed to hold for certain strongly structured algebras such as enveloping algebras of semisimple Lie algebras and Weyl algebras:

> For any k-algebra A of either kind above and any non-trivial finite group G of k-automorphisms of A, the subalgebra A^G of fixed points is not k-isomorphic to A; we shall say that A does not admit Galois embeddings into itself.

Note that this property also holds when A is the tensor algebra of a finite dimensional vector space V and G is a finite subgroup of $GL(V)$. Indeed, in that case, by [Kar] the algebra of invariants is also a tensor algebra, whereas, by [Di Fo], it is finitely generated if and only if G consists of

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scalar multiples of the identity, in which case it is isomorphic to the tensor algebra of $V^{\otimes |G|}$; in particular, the algebra of invariants is isomorphic to the initial tensor algebra if and only if G is trivial. On the other hand, in [A-H-V] a complete classification of finite group actions in the first Weyl algebra and of their invariants is given; in particular, one can observe that the first Weyl algebra admits no Galois embedding into itself. Finally, in [Smi] and later in [Mo-Gu] in a more general form, it is shown that the first Weyl algebra never appears as the invariant subalgebra of a non-perfect finite group of automorphism of *any* k-algebra without zero-divisors. The case of a simple group seems still open. In this paper, we establish the following two theorems.

THEOREM 1. *Let G be a finite group of k-automorphisms of the enveloping algebra $U(\mathbf{g})$ of a semisimple Lie algebra $\mathbf{g}$ over k; if $U(\mathbf{g})^G$ is k-isomorphic to the enveloping algebra of some Lie algebra $\mathbf{g}'$, then $\mathbf{g}' \simeq \mathbf{g}$ and G is trivial. In particular, $U(\mathbf{g})$ does not admit any Galois embedding into itself.*

THEOREM 2. *Let G be a finite group of k-automorphisms of the nth Weyl algebra $A_n(k)$; if $A_n(k)^G$ is k-isomorphic to $A_n(k)$, then G is trivial. In other words, $A_n(k)$ does not admit any Galois embedding into itself.*

We could remark that no linearity is assumed on the action of G; indeed, the usual filtrations of $U(\mathbf{g})$ and $A_n(k)$ are not supposed to be preserved by G and this forces to look for finer automorphism invariants of these algebras. On the one hand, both proofs use general results relating the structure of a ring R to the structure of the fixed subring R^G of a finite automorphism group G. On the other hand, the proof of Theorem 1 is based on very precise information available about primitive ideals of $U(\mathbf{g})$, whereas the proof of Theorem 2 goes by reduction to positive characteristic.

The paper is organized as follows: Theorem 1 is proved in Section 1, whereas Section 2 contains a proof of Theorem 2 as stated above, as well as a shorter proof in the case of a linear action.

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1. ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

1.1. Throughout this section, let k denote an algebraically closed field of characteristic zero, $\mathbf{g}$ a semisimple Lie algebra over k, $U = U(\mathbf{g})$ its enveloping algebra, and G a finite subgroup of $\operatorname{Aut}_k U(\mathbf{g})$. We shall then prove the following.

THEOREM 1. *Assume that $U(\mathbf{g})^G$ is k-isomorphic to the enveloping algebra of some Lie algebra $\mathbf{g}'$; then $\mathbf{g}' \simeq \mathbf{g}$ and G is trivial. In particular, $U(\mathbf{g})$ does not admit any Galois embedding into itself.*

1.2. During the Oberwolfach conference on "Noncommutative Algebra and Representation Theory" (August 16–21, 1993), L. Small informed the first author of some results in the forthcoming paper [Kr–Sm], which imply in particular that the subalgebra of invariants of $U(\mathbf{sl}_2)$ under a non-trivial finite cyclic subgroup of the adjoint group is not even a quotient of the enveloping algebra of any semisimple Lie algebra. This issue was further discussed with L. Le Bruyn, and we would like to thank them both for their interest, which led us to observe that our proof gives, in fact, a stronger result in the case of a subgroup of the adjoint group. Namely, one has the following.

PROPOSITION. *Keep notation as in 1.1 and assume further that G fixes pointwise the center of U (which is the case if G is conjugate in* $\mathrm{Aut}_k\, U(\mathbf{g})$ *to a subgroup of the adjoint group). Then, unless G is trivial, U^G is not even a quotient of the enveloping algebra of any semisimple Lie algebra.*

1.3. Consider the following assertions:

(A) Every irreducible finite dimensional $U(\mathbf{g})$-module remains irreducible by restriction to $U(\mathrm{U})^G$.

(B) The annihilator of each irreducible finite dimensional $U(\mathbf{g})$-module is G-invariant.

Then one has the following.

PROPOSITION. *Assume that assertions* (A) *and* (B) *hold. Then* $G = \{1\}$.

Proof. Let $u \in U$ and $g \in G$. Consider an arbitrary irreducible finite dimensional $U(\mathbf{g})$-module E. Since E remains irreducible by restriction to $U(\mathbf{g})^G$ then, by Jacobson's density theorem, the map $U(\mathbf{g})^G \to \mathrm{End}_k(E)$ is surjective. Therefore there exists $x \in U(\mathbf{g})^G$ such that $u - x \in \mathrm{Ann}\, E$. Since $\mathrm{Ann}\, E$ is G-invariant and x a fixed point of G one obtains $gu - x \in \mathrm{Ann}\, E$, hence $u - gu \in \mathrm{Ann}\, E$. Since E was arbitrary and since the intersection of the annihilators of all irreducible finite dimensional $U(\mathbf{g})$-modules is reduced to $\{0\}$ by a theorem of Harish-Chandra, together with Weyl's complete reducibility theorem (see [Dix, 2.5.7 and 1.6.3]), it follows that $u = gu$. This proves that $G = \{1\}$.

Remark. Note that the finiteness of G was not used in this subsection. Also, let us mention that Theorem 1 was first proved for $\mathbf{g} = \mathbf{sl}_2$ by

M. Chamarie and the first author (unpublished), by using the above argument.

1.4. Let us denote by $Z(A)$ the center of a ring A. Then we have the following.

PROPOSITION. $Z(U^G) = Z(U)^G$.

Proof. First, recall the definition of X-inner automorphisms of a (semiprime) ring; see [Mo 1, Chap. 3]. Then, by [*loc. cit.*, 6.17], the proposition will follow if one checks that $U(\mathbf{g})$ has no X-inner automorphism but the identity. So, let τ be an X-inner automorphism of $U(\mathbf{g})$. Then, by [Mo 2, Proposition 1], τ preserves the canonical filtration of $U(\mathbf{g})$, and induces the identity on the associated graded ring. Hence, there exists a linear form λ on $\mathbf{g}$ such that $\tau(x) = x + \lambda(x)$ for all $x \in \mathbf{g}$, and λ satisfies $\lambda([\mathbf{g}, \mathbf{g}]) = 0$. Indeed, if $x, y \in \mathbf{g}$ then

$$\tau([x, y]) = \tau(xy - yx) = \tau(x)\,\tau(y) - \tau(y)\,\tau(x) = xy - yx = [x, y];$$

hence $\lambda([x, y]) = 0$. But $\mathbf{g} = [\mathbf{g}, \mathbf{g}]$ since $\mathbf{g}$ is semisimple; hence $\lambda = 0$ and $\tau = \mathrm{id}$. This proves the proposition.

1.5. In this subsection we record several facts about finite dimensional irreducible U-modules and their central characters. For short, we denote the center of U simply by Z. Recall first that, since k is algebraically closed, any maximal ideal of Z is the kernel of a (unique) k-algebra homomorphism $Z \to k$, and such a homomorphism is called a central character. Hence there is a bijection between the set $\operatorname{Max} Z$ of maximal ideals of Z, and the set $\operatorname{Char} Z$ of central characters of Z. Also, any $z \in Z$ can be regarded as a regular function on the affine variety $\operatorname{Max} Z$. Moreover, if $\chi \in \operatorname{Char} Z$, and $m = \operatorname{Ker} \chi$ then $\chi(z)$ is precisely the value of z on the point m. For this reason, we shall denote $\chi(z)$ by $\langle z, \chi \rangle$ or $\langle z, m \rangle$.

Now, a U-module is said to admit a central character χ if it is annihilated by the (maximal) ideal $\operatorname{Ker} \chi$ of Z. By Schur's lemma, every (finite dimensional) irreducible U-module E admits a central character, denoted by χ_E. Let us denote by $\mathscr{I}^f(\mathbf{g})$ the set of isomorphy classes of irreducible finite dimensional U-modules, and by Γ the set of all χ_E's, as E varies through $\mathscr{I}^f(\mathbf{g})$. Then one has the following.

PROPOSITION. (a) *For any $\chi \in \Gamma$, there exists a unique element of $\mathscr{I}^f(\mathbf{g})$, denoted by $E(\chi)$, whose central character is χ.*

(b) *The set* $\{\operatorname{Ker}\chi \mid \chi \in \Gamma\}$ *is dense in* Max Z, *for the Zariski topology.*

(c) *There exists a (unique) element* D *of* Z *such that, for all* $\chi \in \Gamma$,

$$\langle D, \chi \rangle = (\dim_k E(\chi))^2.$$

Proof. This is well known: parts (a) and (b) can be found in [Bou, VIII, Section 8, No. 5, Théorème 2 et Corollaires]; part (c) in [*loc. cit.*, Section 9, Exercice 2].

1.6. Recall the definition of Gelfand Kirillov dimension, see [Kr Le], which we shall denote by $d(?)$. Then, for future use, we record here the following.

LEMMA. *Let* A *be a noetherian* k*-algebra,* H *a finite group of automorphisms of* A, *and* J *a two-sided ideal of* A. *Then,* $d(A^H/A^H \cap J) = d(A/J)$.

Proof. This follows from [Mo 1, 5.9; Kr Le, 5.5].

1.7. From now on, we assume that $U(\mathbf{g})^G$ is k-isomorphic to the enveloping algebra of some Lie algebra $\mathbf{g}'$. Then, by the previous lemma, $d(U(\mathbf{g}')) = d(U(\mathbf{g}))$, and by [Kr Le, 6.5 6.9], it follows that $\mathbf{g}'$ is finite dimensional and $\dim_k(\mathbf{g}') = \dim_k(\mathbf{g})$. Moreover, $\mathbf{g}'$ is semisimple, as it follows from the lemma.

LEMMA. *Every finite dimensional* $U(\mathbf{g}')$*-module is the restriction of a finite dimensional* $U(\mathbf{g})$*-module and is, therefore, semisimple. As a consequence,* $\mathbf{g}'$ *is semisimple.*

Proof. Set $U' = U(\mathbf{g})^G \simeq U(\mathbf{g}')$ and let M' be a finite dimensional U'-module. First, since char$(k) = 0$, then, as a U'-bimodule, U' is a direct summand of U. Namely, one has $U = U' \oplus \operatorname{Ker} p$, where p is the projector $p = |G|^{-1} \sum_{g \in G} g$. It follows that M' is a U'-submodule of $M|_{U'}$, where M denotes the left U-module $U \otimes_{U'} M'$. Moreover, a result of Farkas and Snider [Mo 1, 5.9] asserts that U is a finite right U'-module. This gives that M is a finite dimensional, hence completely reducible, U-module. Then, by a result of Lorenz and Passman [*loc. cit.*, 7.6(4)], $M|_{U'}$ is a completely reducible U'-module, and so is its submodule M'. The lemma is proved.

1.8. *Proof of assertion* (A). From now on, we denote $U(\mathbf{g})^G \simeq U(\mathbf{g}')$ simply by U', its center by Z', and introduce notations $\mathcal{I}'(\mathbf{g}')$ and Γ' similar to those for $\mathbf{g}$. Applying Proposition 1.5 to $\mathbf{g}'$ instead of $\mathbf{g}$, we denote, for every $\chi' \in \Gamma'$, by $E'(\chi')$ the corresponding element of $\mathscr{I}^f(\mathbf{g}')$. Also, let D' denote the unique element of Z', similar to the element D of Z.

Now, let $\chi \in \Gamma$ and let $E(\chi)$ be the corresponding element of $\mathscr{I}^f(\mathbf{g})$. Since $Z' \subseteq Z$, by Proposition 1.4, then $E(\chi)$, regarded as a U'-module, admits the central character $\chi' = \chi|_{Z'}$. On the other hand, by Weyl's complete reducibility theorem, $E(\chi)|_{U'}$ is a direct sum of (finite dimensional) irreducible U'-modules, and each of these admits the central character χ'. Hence $\chi' \in \Gamma'$, and $E(\chi)|_{U'}$ is a direct sum of copies of $E'(\chi')$. Let $m(\chi)$ denote the multiplicity of $E'(\chi')$ in $E(\chi)|_{U'}$. Then:

$$\langle D, \chi \rangle = (\dim_k E(\chi))^2 = m(\chi)^2 (\dim_k E'(\chi'))^2 = m(\chi)^2 \langle D', \chi' \rangle. \qquad (*)$$

Also, $\chi' = \chi|_{Z'}$; hence $\langle D', \chi' \rangle$ is nothing but $\langle D', \chi \rangle$. Moreover, $m(\chi)$ is the length of $E(\chi)$ as a $U' = U^G$-module, and by a result of Lorentz and Passmann [Mo 1, Theorem 7.6(3)], one has for all $\chi \in \Gamma$,

$$m(\chi) = \operatorname{length}_{U'}(E(\chi)|_{U'}) \leqslant |G| \operatorname{length}_U(E(\chi)) = |G|.$$

Hence, $m(\chi) \in \{1, ..., |G|\}$ for all $\chi \in \Gamma$. Thus, the element $P = \prod_{i=1}^{|G|} (D - i^2 D')$ of Z satisfies $\langle P, \operatorname{Ker} \chi \rangle = \langle P, \chi \rangle = 0$ for all $\chi \in \Gamma$. Since $\{\operatorname{Ker} \chi \mid \chi \in \Gamma\}$ is a dense subset of Max Z, then P vanishes identically on Max Z, and since the latter is irreducible (Z being a domain), some factor of P also vanishes identically. Hence $D = i^2 D'$ for some $i \in \{1, ..., |G|\}$. Consider now the one-dimensional represenation of U. It certainly restricts to a one-dimensional representation of U', and this gives $i = 1$; hence $D = D'$. It then follows from $(*)$ that $m(\chi) = 1$ for all $\chi \in \Gamma$. This proves assertion (A).

1.9. *Proof of Proposition* 1.2. First, it is easy to see that, if U^G is only assumed to be a quotient of $U(\mathbf{g}')$, with $\mathbf{g}'$ semisimple, then the argument of the previous subsection applies just as well and gives that assertion (A) also holds in this case. Second, under the hypothesis that G fixes pointwise the center of U, it is immediate that assertion (B) is satisfied, since every finite dimensional irreducible U-module is determined by its central character. This proves Proposition 1.2.

1.10. *Towards the proof of assertion* (B). For each $d \in \mathbf{N}^+$, set $\mathscr{I}^f_d(\mathbf{g}) = \{E \in \mathscr{I}^f(\mathbf{g}) \mid \dim_k E = d\}$, define $\mathscr{I}^f_d(\mathbf{g}')$ similarly, and, taking the truth of assertion (A) into account, denote by ϕ_d the map from the former into the latter, which takes $E \in \mathscr{I}^f_d(\mathbf{g})$ to $\phi_d(E) := E|_{U'} \in \mathscr{I}^f_d(\mathbf{g}')$. It is clear from the proof of Lemma 1.7 that every ϕ_d is surjective. For future use, we record this fact as the:

COROLLARY. *ϕ_d is surjective, for every $d \in \mathbf{N}^+$.*

Also, observe that there is a natural action of G on $\mathscr{I}^f(\mathbf{g})$. Indeed, if $E \in \mathscr{I}^f(\mathbf{g})$ and $g \in G$, define the twisted module ${}^gE$ to be the vector space E with the U-module structure given by $u \cdot e = g^{-1}(u)e$, for all $u \in U$, $e \in E$. Then note, on the one hand, that $\operatorname{Ann} {}^gE = g(\operatorname{Ann} E)$, and, on the other hand, that $\phi_d({}^gE) = \phi_d(E)$, where $d = \dim_k E$. Therefore, assertion (B) would follow from the injectivity of the ϕ_d. But it is well known that $\mathscr{I}^f_d(\mathbf{g})$ and $\mathscr{I}^f_d(\mathbf{g}')$ are finite sets (see Lemma 1.16 below); therefore, in view of the previous corollary, injectivity will follow if we prove that, for every $d \in \mathbf{N}^+$, $\mathscr{I}^f_d(\mathbf{g})$ and $\mathscr{I}^f_d(\mathbf{g}')$ have the same cardinality. It is certainly enough to prove that $\mathbf{g} \simeq \mathbf{g}'$, and this is what we shall do.

1.11. Let $\mathbf{h}$ be a Cartan subalgebra of $\mathbf{g}$, R the root system of $(\mathbf{g}, \mathbf{h})$, Δ a basis of R, and define $\mathbf{h}' \subset \mathbf{g}'$, R', and Δ' similarly. Then one has the lemma.

LEMMA. (a) $\dim_k \mathbf{g} = d(U) = d(U') = \dim_k \mathbf{g}'$

(b) $|\Delta| = d(Z) = d(Z') = |\Delta'|$

(c) $|R| = |R'|$.

Proof. Since $\dim_k \mathbf{g} = |\Delta| + |R|$, and similarly for $\mathbf{g}'$, assertion (c) is a consequence of assertions (a) and (b), which themselves follow from Lemma 1.6, together with [Kr-Le, 6.9; Dix, 7.3.8].

1.12. *Primitive ideals.* Keep the notations of 1.11, and, in addition, introduce: $\mathbf{g} = \mathbf{h} \oplus (\bigoplus_{\alpha \in R} \mathbf{g}_\alpha)$ the corresponding weight space decompositon, $R^\vee = \{H_\alpha \mid \alpha \in R\} \subset \mathbf{h}$ the set of coroots, W the Weyl group, $R^+ = R \cap \mathbf{N}\Delta$ the set of positive roots corresponding to Δ, ρ the half-sum of the elements of R^+, and w_0 the unique element of W such that $w_0(R^+) = -R^+$. Recall that to each $\alpha \in \Delta$ is associated an element of W, the reflection s_α. Then, for each subset S of Δ, set $R_S = R \cap \mathbf{Z}S$, $R_S^+ = R_S \cap R^+$, let W_S be the subgroup of W generated by the reflections s_α, where $\alpha \in S$, and let w_S denote the unique element of W_S such that $w_S(R_S^+) = -R_S^+$.

Now, set $\mathbf{n} = \bigoplus_{\alpha \in R^+} \mathbf{g}_\alpha$ and $\mathbf{b} = \mathbf{h} \oplus \mathbf{n}$. Then $\mathbf{b}$ is a Borel subalgebra of $\mathbf{g}$ and $\mathbf{n}$ an ideal of $\mathbf{b}$. Let $\lambda \in \mathbf{h}^*$. Then λ defines a one-dimensional representation of $\mathbf{h}$. Since $\mathbf{b}/\mathbf{n} \simeq \mathbf{h}$, then λ also defines a one-dimensional representation of $\mathbf{b}$, which we shall denote by k_λ. One then defines the Verma module $M(\lambda) = U(\mathbf{g}) \otimes_{U(\mathbf{b})} k_\lambda$. By [Dix, 7.1.11–7.1.13], $M(\lambda)$ has a unique simple quotient, denoted by $L(\lambda)$, and $L(\lambda)$ is characterized by the existence of a non-zero vector $v \in L(\lambda)$ such that $\mathbf{n}v = 0$ and $(h - \lambda(h))v = 0$ for all $h \in \mathbf{h}$. (Beware the change of notation: our $M(\lambda)$ is denoted $M(\lambda + \rho)$

in *loc. cit.*). Since the unique one-dimensional $\mathbf{g}$-module is annihilated by $[\mathbf{g}, \mathbf{g}] = \mathbf{g}$, and in particular by $\mathbf{b}$, this module is indeed $L(0)$. Also, for every $\lambda \in \mathbf{h}^*$, set $I(\lambda) = \mathrm{Ann}_U L(\lambda)$. Recall the definition of the dot-action of W on $\mathbf{h}^*$: for any $w \in W$, $\lambda \in \mathbf{h}^*$, $w \cdot \lambda = w(\lambda + \rho) - \rho$. Then one has the following.

PROPOSITION. *Let χ_0 denote the central character of $L(0)$, and set $I = U(\mathrm{Ker}\, \chi_0)$. Then*:

(a) *I is left invariant by every automorphism of U.*

(b) *I is completely prime, and $I = I(w_0 \cdot 0)$.*

(c) *The prime ideals of U, minimal among those strictly containing I, are exactly the $I(s_\alpha w_0 \cdot 0) := I_\alpha$, where $\alpha \in \Delta$.*

(d) *For any subset S of Δ, one has $\sum_{\alpha \in S} I(s_\alpha w_0 \cdot 0) = I(w_S w_0 \cdot 0) := I_S$; this ideal is completely prime, and $d(U/I_S) = |R| - |R_S|$. Also, $d(U/I) = |R|$.*

Proof. Since $\mathbf{g}$ has a unique one-dimensional representation, then U has a unique two-sided ideal of codimension one, which we shall denote by U_+. Clearly, U_+ is left invariant by every automorphism of U; hence so is $\mathrm{Ker}\, \chi_0 = U_+ \cap Z$. This proves assertion (a). Assertion (b) follows from [Dix, 8.4.3–8.4.4, 7.4.7, and 7.6.24]. Assertion (c) and the first part of assertion (d) follow from [Duf, Corollaire 2 de la Proposition 10, Proposition 12]. The second part of assertion (d) follows from [Jan, 15.3(5) and 15.6]. Finally, the assertions concerning Gelfand–Kirillov dimension follow from [*loc. cit.*, 15.3(1) and 10.9].

1.13. Set $\chi'_0 = \chi_0|_{Z'}$, and $I' = U'(\mathrm{Ker}\, \chi'_0)$. Since χ'_0 is the central character of $L(0)|_{U'}$, the unique one-dimensional U'-module, then Proposition 1.12 also applies to I'. In particular, the prime ideals of U', minimal among those strictly containing I', are denoted by $I'_{\alpha'}$, where α' runs through Δ'. Then we have the following.

PROPOSITION. (a) *$I \cap U' = I'$.*

(b) *There exists a bijection $\varphi: \Delta \to \Delta'$ such that, for all $\alpha \in \Delta$, $I_\alpha \cap U' = I'_{\varphi(\alpha)}$.*

(c) *I_α is G-invariant, for every $\alpha \in \Delta$.*

Proof. Clearly, $(I \cap U') \supseteq I'$. Moreover, by 1.6, 1.11(c), and the last assertion of 1.12 applied to both U and U', one has

$$d(U'/U' \cap I) = d(U/I) = |R| = |R'| = d(U'/I').$$

Since U'/I' is prime noetherian, this gives $U' \cap I = I'$, by [Kr–Le, 3.15] together with Goldie's theorem (see, e.g., [Dix, 3.5.10]). Thus, since I is

G-stable by Proposition 1.12(a), setting $A = U/I$ and $A' = U'/I'$ one then has $A^G = A'$. Recall then that, by a result of Montgomery [Mo 3, 4.2 and 3.6], there exists a bijective, order preserving, correspondence between G-orbits in Spec A and certain equivalence classes in Spec A'. In our case, a part of this correspondence can be expressed in a simple manner, since some equivalence classes in Spec A' are trivial. Indeed, for every minimal non-zero prime ideal J of A, the ideal $J \cap A'$ is completely prime since J is so. It then follows from [*loc. cit.*] that the map $\phi : J \mapsto J \cap A'$ is a surjective map from the set of minimal non-zero primes of A onto the set of minimal non-zero primes of A', and every fiber of ϕ is a G-orbit. But we saw already that A and A' have the same number of minimal non-zero prime ideals, namely $|\Delta| = |\Delta'|$. This simultaneously gives that ϕ is bijective and hence induces a bijection $\varphi : \Delta \to \Delta'$ such that $U' \cap I_\alpha = I'_{\varphi(\alpha)}$ for all $\alpha \in \Delta$, and that all I_α, where $\alpha \in \Delta$, are G-invariant. The proposition is proved.

1.14. *Coxeter graphs.* For any pair α, β of elements of Δ, denote by $m_{\alpha\beta}$ the order of the element $s_\alpha s_\beta$ of W. Then recall (see [Bou, IV, Section 1, No. 9]) that the Coxeter graph of $\mathbf{g}$ is the labelled graph defined as follows: its set of vertices is Δ, and $\{\alpha, \beta\}$ is an edge if and only if $m_{\alpha\beta} \geqslant 3$, in which case the edge $\{\alpha, \beta\}$ carries the label $m_{\alpha\beta}$.

LEMMA. *One has* $2m_{\alpha\beta} = |R_{\{\alpha,\beta\}}|$, *for every pair of elements* α, β *in* Δ.

Proof. Let $W_{\{\alpha,\beta\}}$ be the subgroup of W generated by s_α and s_β. Recall that the length $l(w)$ of an element $w \in W_{\{\alpha,\beta\}}$ is the smallest integer $q \geqslant 0$ such that w can be written as a product of q elements of the set $\{s_\alpha, s_\beta\}$. By [Bou, IV, Section 1, No. 2, Remarque], $W_{\{\alpha,\beta\}}$ contains a unique element of maximal length, denoted by $w_{\{\alpha,\beta\}}$, and $l(w_{\{\alpha,\beta\}}) = m_{\alpha\beta}$. On the other hand, by [Bou, VI, Section 1, No. 6, Corollaire 3 de la Proposition 17], one has $l(w_{\{\alpha,\beta\}}) = |R^+_{\{\alpha,\beta\}}|$. The lemma follows.

PROPOSITION. *The bijection* $\varphi : \Delta \to \Delta'$ *of Proposition* 1.13(b) *is an isomorphism of Coxeter graphs.*

Proof. Let $\{\alpha, \beta\}$ be a pair of elements of Δ. Since I_α and I_β are G-stable by Proposition 1.13 and since the functor $M \mapsto M^G$ is exact (recall char$(k) = 0$) one has $(I_\alpha + I_\beta)^G = I_\alpha^G + I_\beta^G = I'_{\varphi(\alpha)} + I'_{\varphi(\beta)}$. Therefore, by Lemma 1.6 and by Proposition 1.12(d) applied to U and to U', we obtain

$$\begin{aligned}|R| - |R_{\{\alpha,\beta\}}| &= d\left(\frac{U}{I_\alpha + I_\beta}\right)\\ &= d\left(\frac{U'}{I'_{\varphi(\alpha)} + I'_{\varphi(\beta)}}\right) = |R'| - |R'_{\{\varphi(\alpha),\varphi(\beta)\}}|.\end{aligned}$$

Since $|R| = |R'|$ by Lemma 1.11, it follows that $|R_{\{\alpha,\beta\}}| = |R'_{\{\varphi(\alpha),\varphi(\beta)\}}|$. Applying the previous lemma on both sides, we obtain $m_{\alpha\beta} = m_{\varphi(\alpha)\varphi(\beta)}$. This proves the proposition.

1.15. Recall (see, e.g., [Bou, VIII, Section 4, No. 4, Théorème 2(iii)]) that a semisimple Lie algebra over an algebraically closed field of characteristic zero is determined up to isomorphism by its Dynkin graph (defined, e.g., in [*loc. cit.*, VI, Section 4, No. 2]). In particular, the latter determines the Coxeter graph. Conversely, it is well known (see, e.g., [*loc. cit.*, Section 4, Théorèmes 1 and 3]) that two connected Dynkin graphs having the same Coxeter graph are isomorphic, unless they are of types B_n and C_n, for some $n \geqslant 3$. This leads us to introduce the following notations. First, denote by $\mathscr{D}(\mathbf{g})$ the Dynkin graph of $\mathbf{g}$ and by X the set of connected components of $\mathscr{D}(\mathbf{g})$. Then recall (see [*loc. cit.*]) that the set of vertices of $\mathscr{D}(\mathbf{g})$ may be identified with Δ, the set of simple roots. Making this identification, let us then denote, for every $\mathscr{C} \in X$, by $\mathbf{g}_{\mathscr{C}}$ the subalgebra of $\mathbf{g}$ generated by the subspaces $\mathbf{g}_{\pm\alpha}$, where $\alpha \in \mathscr{C}$. Then $\mathbf{g}_{\mathscr{C}}$ is a simple Lie algebra, and $\mathscr{C}$ is its Dynkin graph. Moreover, $\mathbf{g}$ is the direct product of the $\mathbf{g}_{\mathscr{C}}$, as $\mathscr{C}$ runs through X. Define similarly X', and the subalgebras $\mathbf{g}'_{\mathscr{C}'}$ of $\mathbf{g}'$, for $\mathscr{C}' \in X'$. Also, for every integer $n \geqslant 3$, let us denote by:

$\mathbf{b}_n$ (resp. $\mathbf{c}_n$) the simple Lie algebra of type B_n (resp. C_n).

r_n (resp. s_n) the number of connected components of $\mathscr{D}(\mathbf{g})$ of type B_n (resp. C_n).

$X_{\geqslant n}$ (resp. X_n) the set of connected components of $\mathscr{D}(\mathbf{g})$ which are of type B_p or C_p, with $p \geqslant n$ (resp. $p = n$).

$$\mathbf{p}_n = \prod_{\mathscr{C} \notin X_{\geqslant n}} \mathbf{g}_{\mathscr{C}}, \quad \mathbf{g}_n = \prod_{\mathscr{C} \in X_n} \mathbf{g}_{\mathscr{C}}, \quad \mathbf{q}_n = \prod_{\mathscr{C} \in X_{\geqslant n+1}} \mathbf{g}_{\mathscr{C}}.$$

Then, one has $\mathbf{g} \simeq \mathbf{p}_n \times \mathbf{g}_n \times \mathbf{q}_n$ and $\mathbf{g}_n \simeq (\mathbf{b}_n)^{r_n} \times (\mathbf{c}_n)^{s_n}$.

Define similarly r'_n, s'_n, $\mathbf{p}'_n, \mathbf{g}'_n$, and $\mathbf{q}'_n$. Then, with these notations, Proposition 1.14 has the following consequence.

COROLLARY. *One has* $\mathbf{p}_3 \simeq \mathbf{p}'_3$, *and* $r_n + s_n = r'_n + s'_n$ *for every* $n \geqslant 3$.

By induction on n, we are going to prove that $\mathbf{p}_n \simeq \mathbf{p}'_n$, for all $n \geqslant 3$. We shall need several lemmas.

1.16. Set $\mathscr{P}^+ = \{\lambda \in \mathbf{h}^* \mid \lambda(H_\alpha) \in \mathbf{N}\ \forall \alpha \in \Delta\}$, and let $\{\omega_\alpha\}_{\alpha \in \Delta}$ denote the basis of $\mathbf{h}^*$ dual to the basis $\{H_\alpha\}_{\alpha \in \Delta}$ of $\mathbf{h}$; the ω_α are called the fundamental weights, they form an $\mathbf{N}$-basis of $\mathscr{P}^+$. Indeed, for any $\lambda \in \mathscr{P}^+$ and $\alpha \in \Delta$, set $\lambda_\alpha = \lambda(H_\alpha) \in \mathbf{N}$; then $\lambda = \sum_{\alpha \in \Delta} \lambda_\alpha \omega_\alpha$. Define a partial order $\leqslant$ on $\mathscr{P}^+$ as follows: $\lambda \leqslant \mu$ if and only if $\lambda_\alpha \leqslant \mu_\alpha$ for all $\alpha \in \Delta$. Also, set $\lambda < \mu$ if $\lambda \leqslant \mu$ and $\lambda \neq \mu$. Then, one has the following.

LEMMA. (a) *The map* $\lambda \mapsto [L(\lambda)]$ *is a bijection from* $\mathscr{P}^+$ *to* $\mathscr{I}'(\mathbf{g})$.

(b) $\mathscr{I}'_d(\mathbf{g})$ *is a finite set, for every* $d \in \mathbf{N}^+$.

(c) *Let* $\lambda, \mu \in \mathscr{P}^+$. *If* $\lambda \prec \mu$, *then* $\dim_k L(\lambda) < \dim_k L(\mu)$.

Proof. Assertion (a) follows from [Dix, 7.2.6], whereas assertions (b) and (c) will follow from the Weyl character formula: for all $\lambda \in \mathscr{P}^+$,

$$\dim_k L(\lambda) = \prod_{\alpha \in R^+} \frac{(\rho + \lambda)(H_\alpha)}{\rho(H_\alpha)}.$$

Indeed, recall first that $\rho(H_\alpha) = 1$ for all $\alpha \in \Delta$, whence $\rho(H_\alpha)$ and $(\lambda + \rho)(H_\alpha)$ are positive integers for all $\alpha \in R^+$. Now, if $\dim_k L(\lambda) = d$, then $\prod_{\alpha \in R^+} (\rho + \lambda)(H_\alpha) = d \prod_{\alpha \in R^+} \rho(H_\alpha) := d' \in \mathbf{N}^+$. This gives, for instance, that $\lambda_\alpha \leqslant d' - 1$ for all $\alpha \in \Delta$, and assertion (b) follows.

Cosider now assertion (c). First, for every $\beta \in R^+$, one has $H_\beta = \sum_{\alpha \in \Delta} c_{\beta\alpha} H_\alpha$ for some $c_{\beta\alpha} \in \mathbf{N}$; hence, if $\lambda, \mu \in \mathscr{P}^+$ and $\lambda \preccurlyeq \mu$ then

$$(\rho + \lambda)(H_\beta) = \sum_{\alpha \in \Delta} c_{\beta\alpha}(\rho_\alpha + \lambda_\alpha) \leqslant \sum_{\alpha \in \Delta} c_{\beta\alpha}(\rho_\alpha + \mu_\alpha) = (\rho + \mu)(H_\beta).$$

Moreover, if $\lambda \prec \mu$ then the above inequality is strict for at least one $\beta \in \Delta$, and assertion (c) follows.

1.17. LEMMA. *Let* $n \geqslant 3$. *Then*:

(a) *Every non-trivial irreducible representation of* $\mathbf{b}_n$ *has dimension* $\geqslant 2n+1$, *and there is a unique, up to isomorphism, such representation of dimension* $2n+1$.

(b) *Every non-trivial irreducible representation of* $\mathbf{c}_n$ *has dimension* $\geqslant 2n$, *and there is a unique, up to isomorphism, such representation of dimension* $2n$, *but none of dimension* $2n+1$.

Proof. In both cases, we fix numberings $\alpha_1, \ldots, \alpha_n$ and, correspondingly, $\omega_1, \ldots, \omega_n$ of the simple roots and fundamental weights, as in [Bou, VI, Planches II III]. Then, one has $\dim_k L(\omega_1) = 2n+1$ for $\mathbf{b}_n$, whereas $\dim_k L(\omega_1) = 2n$ for $\mathbf{c}_n$. Hence, by Lemma 1.16, it is enough to prove, in both cases, that $\dim_k L(\lambda) \geqslant 2n+2$ when λ is either $2\omega_1$, or ω_i for some $i \geqslant 2$.

Consider $\mathbf{b}_n$. By [Bou, VIII, Section 13, No. 2], one has $L(\omega_i) \simeq \bigwedge^i L(\omega_1)$ for $2 \leqslant i \leqslant n-1$, and $\dim_k L(\omega_n) = 2^n$. Hence, one has

$$\dim_k L(\omega_i) = \binom{2n+1}{i} > 2n+1 \qquad \text{for} \quad 2 \leqslant i \leqslant n-1.$$

Taking the hypothesis $n \geqslant 3$ into account, one also obtains

$$\dim_k L(\omega_n) = 2^n = \sum_{j=0}^{n} \binom{n}{j} \geqslant 1 + n + \binom{n}{2} + 1 \geqslant 2n + 2.$$

Also, direct computations, using Weyl's character formula, together with the lists of positive roots in [Bou, VI, Planches II–III], give, on the one hand, that

$$\dim_k L(2\omega_1) = \begin{cases} n(2n+3) & \text{for } \mathbf{b}_n \\ n(2n+1) & \text{for } \mathbf{c}_n \end{cases}$$

and, on the other hand, that, for $\mathbf{c}_n$ and any $i \in \{2, ..., n\}$,

$$\dim_k L(\omega_i) = (2n+1) \times \prod_{j=0}^{i-3} \frac{2n-j}{2n-j-i} \times \frac{1}{2n+1-i} \binom{2n+1-i}{i}$$

with the convention that the factor in the middle equals 1 if the indexing set is empty, that is, if $i = 2$. Otherwise, if $i \geqslant 3$, this factor is a rational number > 1. Moreover, since binomial coefficients are unimodal, then the last factor is also a rational number > 1, unless $i = 2n - i$, that is, unless $i = n$. Since $n \geqslant 3$, the conditions $i = 2$ and $i = n$ cannot simultaneously be realized, hence $\dim_k L(\omega_i) > 2n + 1$. The lemma is proved.

1.18. For each $\mathscr{C} \in X$, the linear span of the H_α, where $\alpha \in \mathscr{C}$, is a Cartan subalgebra of $\mathbf{g}_{\mathscr{C}}$, denoted by $\mathbf{h}_{\mathscr{C}}$; we set $\mathscr{P}_{\mathscr{C}}^+ = \{\lambda \in \mathbf{h}_{\mathscr{C}}^* \mid \lambda(H_\alpha) \in \mathbf{N}\ \forall \alpha \in \mathscr{C}\}$. Then $\mathbf{h}, \mathbf{h}^*$, and $\mathscr{P}^+$ are respectively the direct product of the $\mathbf{h}_{\mathscr{C}}, \mathbf{h}_{\mathscr{C}}^*$, and $\mathscr{P}_{\mathscr{C}}^+$, as $\mathscr{C}$ runs through X.

Now, consider $\lambda = (\lambda_{\mathscr{C}})_{\mathscr{C} \in X}$ in $\mathbf{h}^*$. For every $\mathscr{C} \in X$, one defines the irreducible $U(\mathbf{g}_{\mathscr{C}})$-module $L_{\mathscr{C}}(\lambda_{\mathscr{C}})$ as in 1.12. Moreover, if $\lambda \in \mathscr{P}^+$ then every $\lambda_{\mathscr{C}}$ belongs to $\mathscr{P}_{\mathscr{C}}^+$, so that $L_{\mathscr{C}}(\lambda_{\mathscr{C}})$ is finite dimensional, by Lemma 1.16(a) applied to $\mathbf{g}_{\mathscr{C}}$. Finally, since $\mathbf{g} = \prod_{\mathscr{C}} \mathbf{g}_{\mathscr{C}}$, then every $L_{\mathscr{C}}(\lambda_{\mathscr{C}})$ is a $\mathbf{g}$-module and so is their tensor product. Then one has the following (well-known) lemma (see, e.g., [Bou, VIII, Section 7, Exercice 2]).

LEMMA. *For every* $\lambda = (\lambda_{\mathscr{C}})_{\mathscr{C} \in X}$ *in* $\mathscr{P}^+$, *one has* $L(\lambda) \simeq \bigotimes_{\mathscr{C} \in X} L_{\mathscr{C}}(\lambda_{\mathscr{C}})$.

Proof. Set $M = \bigotimes_{\mathscr{C}} L_{\mathscr{C}}(\lambda_{\mathscr{C}})$. Clearly, M contains a non-zero vector annihilated by $\mathbf{n}$ and by $h - \lambda(h)$, for all $h \in \mathbf{h}$. Hence, by [Dix, 7.1.13], it is enough to check that M is irreducible. Since M is finite dimensional, as we already observed, it is completely reducible. Therefore, irreducibility will follow if we check that $\operatorname{End}_{\mathbf{g}} M = k$. But, since each $\mathbf{g}_{\mathscr{C}}$ acts trivially on all factors of the tensor product, except the one corresponding to $\mathscr{C}$, it easily follows that $\operatorname{End}_{\mathbf{g}} M \simeq \bigotimes_{\mathscr{C}} \operatorname{End}_{\mathbf{g}_{\mathscr{C}}} L_{\mathscr{C}}(\lambda_{\mathscr{C}}) \simeq k$. This proves the lemma.

1.19. *Completion of the proof.* We can now finish the proof of Theorem 1. Recall the notations of Subsection 1.15 and let $n \geqslant 3$. By induction hypothesis, we may assume that $\mathbf{p}_n \simeq \mathbf{p}'_n$. If follows from Lemmas 1.18 and 1.17 that

$$|\mathscr{I}^{\,t}_{2n}(\mathbf{g})| = |\mathscr{I}^{\,t}_{2n}(\mathbf{p}_n)| + |\mathscr{I}^{\,t}_{2n}(\mathbf{g}_n)| = |\mathscr{I}^{\,t}_{2n}(\mathbf{p}_n)| + s_n$$

and

$$|\mathscr{I}^{\,t}_{2n+1}(\mathbf{g})| = |\mathscr{I}^{\,t}_{2n+1}(\mathbf{p}_n)| + |\mathscr{I}^{\,t}_{2n+1}(g_n)| = |\mathscr{I}^{\,t}_{2n+1}(\mathbf{p}_n)| + r_n.$$

Similarly,

$$|\mathscr{I}^{\,t}_{2n}(\mathbf{g}')| = |\mathscr{I}^{\,t}_{2n}(\mathbf{p}'_n)| + s'_n, \qquad |\mathscr{I}^{\,t}_{2n+1}(\mathbf{g}')| = |\mathscr{I}^{\,t}_{2n+1}(\mathbf{p}'_n)| + r_n.$$

On the other hand, by Corollary 1.10, one has $|\mathscr{I}^{\,t}_d(\mathbf{g})| \geqslant |\mathscr{I}^{\,t}_d(\mathbf{g}')|$ for all $d \in \mathbf{N}^+$ and, in particular, for $d = 2n$ and $2n+1$. Taking the isomorphism $\mathbf{p}_n \simeq \mathbf{p}'_n$ into account, one therefore obtains $s_n \geqslant s'_n$ and $r_n \geqslant r'_n$. Since $r_n + s_n = r'_n + s'_n$ by Corollary 1.15, this gives $s_n = s'_n$ and $r_n = r'_n$; hence $\mathbf{p}_{n+1} \simeq \mathbf{p}'_{n+1}$. Since $\mathbf{g} = \mathbf{p}_t$ and $\mathbf{g}' = \mathbf{p}'_t$ for t large enough, one therefore obtains $\mathbf{g} \simeq \mathbf{g}'$. As noted in Subsection 1.10, this completes the proof of Theorem 1.

2. Weyl Algebras

2.1. In this section, let K denote a field of characteristic zero, and let G be a finite group of K-automorphisms of the nth Weyl algebra $A_n(K)$. We shall then prove the following.

Theorem 2. *If $A_n(K)^G$ is K-isomorphic to $A_n(K)$, then G is trivial. In other words, $A_n(K)$ does not admit any Galois embedding into itself.*

2.2. Recall that the nth Weyl algebra $A_n(K)$ is the K-algebra with generators $p_1, q_1, ..., p_n, q_n$ and relations: $[p_i, p_j] = [q_i, q_j] = 0$, and $[p_i, q_j] = \delta_{ij}$. It is a simple noetherian domain, and its only units are the non-zero elements of K (see, e.g., [Dix, 4.6.3–4.6.6]).

2.3. *Separability.* Consider an inclusion of rings $R \subset S$. After [L V V, II.5.1], one says that S is separable over R if the S-bimodule map $S \otimes_R S \to S$, $s \otimes s' \mapsto ss'$, admits a splitting. This generalizes the classical notion of separable extension when R is commutative (see, e.g., [De In]).

2.4. Let S be a ring, H a finite group of automorphisms of S, and $R=S^H$. For every $h\in H$, there exists an S-bimodule map $\varphi_h : S\otimes_R S\to {}_hS_1$, $s\otimes s'\mapsto h(s)s'$, where ${}_hS_1$ denotes S, regarded as a S-bimodule for the actions: $s\cdot t=h(s)t$ and $t\cdot s'=ts'$ for all $s, s'\in S$, $t\in {}_hS_1$. Note that φ_1 is precisely the map considered in the previous subsection. Then, one has the following.

PROPOSITION. *Keep the above notations and assume that there exist elements $a_1, b_1, ..., a_m, b_m$ in S such that $\sum_{i=1}^m g(a_i)\, h(b_i)=\delta_{g,h}$ for all $g, h\in H$. Then the map $\bigoplus_h \varphi_h$: $S\otimes_R S\to \bigoplus_h {}_hS_1$ is an isomorphism of S-bimodules. In particular, S is separable over R.*

Proof. Set $\phi=\bigoplus_h \varphi_h$. It is clear that ϕ is an homomorphism of S-bimodules. On the other hand, define $\psi : \bigoplus_h {}_hS_1\to S\otimes_R S$ as follows: if $u_h\in {}_hS_1$ then $\psi(u_h)=\sum_{i=1}^m a_i\otimes h(b_i)u_h$. One then checks that

$$(\phi\circ\psi)(u_h)=\sum_{g\in H}\sum_i g(a_i)\, h(b_i)\, u_h=\sum_g \delta_{g,h}u_h=u_h$$

and also, if $x, y\in S$,

$$\begin{aligned}(\psi\circ\phi)(x\otimes y)&=\psi\left(\sum_{h\in H} h(x)y\right)\\
&=\sum_h\sum_i a_i\otimes h(b_i)\, h(x)\, y\\
&=\sum_i a_i\otimes\sum_h h(b_ix)y\\
&=\sum_i a_i\left(\sum_h h(b_ix)\right)\otimes y \qquad \left(\text{since} \sum_h h(b_ix)\in R\right)\\
&=\sum_h\sum_i a_ih(b_i)\, h(x)\otimes y\\
&=\sum_h \delta_{1,h}h(x)\otimes y\\
&=x\otimes y.\end{aligned}$$

Therefore ψ is the inverse of ϕ. This proves the proposition.

2.5. Now, set $A=A_n(K)$, and recall that G is a finite group of K-automorphisms of A. Then, one has the following.

PROPOSITION. *There exist elements $a_1, b_1, ..., a_m, b_m$ in A such that $\sum_{i=1}^m g(a_i)\, h(b_i)=\delta_{g,h}$ for all $g, h\in G$.*

Proof. Since the only invertible elements of A are the nonzero scalars, which are central, then A has no inner automorphism, but the identity. Therefore, it follows from [Mo1, 2.3] that the skew group ring $S = A * G$ is a simple ring. Recall that S is a free left A-module with basis e_g, for $g \in G$, and multiplication is given by $(ae_g)(a'e_h) = ag^{-1}(a')e_{gh}$ for all $a, a' \in A$, $g, h \in G$. Now, consider the element $f = \sum_h e_h$ of S. Note that $e_g f = fe_g = f$ for all $g \in G$; hence $SfS = AfA$. Therefore, since $f \neq 0$ and S is simple, there exist elements $a_1, b_1, ..., a_m, b_m$ in A such that $\sum_{i=1}^m a_i f b_i = 1$. Then

$$1 = \sum_i \sum_h a_i e_h b_i = \sum_i \sum_h a_i h^{-1}(b_i) e_h = \sum_h \left(\sum_i a_i h^{-1}(b_i) \right) e_h$$

and, therefore, $\Sigma_i a_i h^{-1}(b_i) = \delta_{1,h}$; applying another element $g \in G$ to this equality, we get $\sum_i g(a_i)\, gh^{-1}(b_i) = \delta_{1,h} = \delta_{g, gh^{-1}}$. Up to the change of variables $h' = gh^{-1}$, this is the sought for equality.

2.6. From now on, we make the assumption that $A_n(K)^G \simeq A_n(K)$, and we shall obtain the conclusion that $|G| = 1$ by carrying the whole situation over a finite field. First, one has the following.

PROPOSITION. *There exists a finitely generated* $\mathbf{Z}$*-subalgebra* Λ *of* K, *such that, first the subalgebra* $A_n(\Lambda)$ *of* $A_n(K)$ *is* G*-invariant and, second, for any maximal ideal* m *of* Λ, *with residue field* $k = \Lambda/m$, *the following hold*:

(a) *The action of* G *on* $A_n(\Lambda)/mA_n(\Lambda) \simeq A_n(k)$ *is faithful.*

(b) $A_n(k)$ *is separabe over* $A_n(k)^G$.

(c) $A_n(k)^G$ *is* k*-isomorphic to* $A_n(k)$.

Proof. Being finitely generated over K, $A_n(K)$ is the union of its subalgebras $A_n(\Lambda)$, where Λ runs through the finitely generated subrings of K. Since G is finite, it follows that there exists a subring Λ_0 as above, such that $A_n(\Lambda_0)$ is G-stable. Up to enlarging Λ_0, we may assume that the elements a_i, b_i of 2.5 belong to Λ_0, and also that $|G|^{-1} \in \Lambda_0$. Thanks to the latter assumption, one can consider the projector $p = |G|^{-1} \sum_g g$, and this has the consequence that, for any over-ring Λ of Λ_0, one has

$$A_n(\Lambda)^G = p(A_n(\Lambda)) = p(\Lambda A_n(\Lambda_0)) = \Lambda p(A_n(\Lambda_0)) = \Lambda A_n(\Lambda_0)^G. \qquad (*)$$

Moreover, under the same assumption, $A_n(\Lambda_0)^G$ is a finitely generated Λ_0-algebra, by [Mo Sm, Theorem 1]. On the other hand, by our main assumption, $A_n(K)^G$ is also a Weyl algebra, say with generators P_i, Q_i; hence there exists a finitely generated subring Λ of K, containing Λ_0, such that the generators of $A_n(A_0)^G$ belongs to the subalgebra $A\{P_i, Q_i\}$. Taking $(*)$ into account, one then obtains that $A_n(A)^G$ equals $A\{P_i, Q_i\}$ and is, therefore, A-isomorphic to $A_n(A)$.

Let now $m \in \operatorname{Max} A$ and set $k = A/m$. Then $A_n(A)/mA_n(A) \simeq A_n(k)$; also, $mA_n(A)$ is G-stable and, since $|G|^{-1} \in A$, then

$$A_n(k)^G \simeq (A_n(A)/mA_n(A))^G \simeq A_n(A)^G/mA_n(A)^G$$
$$\simeq A_n(A)/mA_n(A) \simeq A_n(k).$$

Finally, the identities $\sum_i g(\bar{a}_i)\, h(\bar{b}_i) = \delta_{g,h}$ in $A_n(k)$ give both the faithfulness of the action of G and the separability over $A_n(k)^G$.

2.7. From now on, we fix a maximal ideal m of A and set $k = A/m$. Then k is a finite field, say of characteristic l. Note that l is prime to $|G|$, since $|G|^{-1} \in A$. Set $A_n(k) = B$ and $A_n(k)^G = B'$ and denote by Z and Z' their respective centers. Then, one has the following.

PROPOSITION. *G acts faithfully on Z.*

Proof. Denote by $\bar{k}$ the algebraic closure of k, set $\bar{B} = B \otimes_k \bar{k}$, and define $\bar{Z}$ and $\bar{B}'$ similarly. Then $\bar{B} \simeq A_n(\bar{k})$; its center is $\bar{Z}$. The action of G extends to $\bar{B}$, and $\bar{B}^G$ equals $\overline{B'}$, hence is $\bar{k}$-isomorphic to $\bar{B}$. Now, by [Rev], $\bar{B}$ is an Azumaya algebra, of rank l^{4n} over its center $\bar{Z}$; in particular, every simple quotient algebra of $\bar{B}$ has dimension l^{4n} over $\bar{k}$, and the same is true for $\overline{B'}$, which is $\bar{k}$-isomorphic to $\bar{B}$. It follows that, for every maximal ideal J of $\bar{B}$, the inclusion $\bar{B}^G/(\bar{B}^G \cap J) \subsetneq \bar{B}/J$ is in fact an equality.

Assume now that $\sigma \in G$ acts trivially on Z, and let $\bar{\sigma} = \sigma \otimes 1$ denote its extension to $\bar{B}$. Let $J \in \operatorname{Max} \bar{B}$; since $\bar{B}$ is Azumaya, J is generated by its intersection with $\bar{Z}$ and is, therefore, $\bar{\sigma}$-stable. Hence $\bar{\sigma}$ acts on $\bar{B}/J$, and this action is trivial, since $\bar{B}/J = \bar{B}^G/(\bar{B}^G \cap J)$. Therefore, for every $b \in \bar{B}$, $b - \bar{\sigma}(b) \in J$. Since J is arbitrary and since the intersection of all maximal ideals of $\bar{B}$ is reduced to $\{0\}$, this gives $\bar{\sigma} = \mathrm{id}$. It follows that $\sigma = \mathrm{id}$; the proposition is proved.

COROLLARY. *One has $Z' = Z^G$.*

Proof. Certainly, any X-inner automorphism of B acts trivially on Z; therefore the previous proposition says that B has no such automorphism but the identity. The corollary then follows from [Mo 1, 6.17].

2.8. Combining 2.6 and 2.7, we are now able to obtain the following.

PROPOSITION. (a) *Z is separable over Z'.*

(b) *For every $m \in \operatorname{Max} Z$ of codimension 1, the orbit Gm has cardinality $|G|$.*

Proof. By Proposition 2.6(b), B is separable over B', whereas B' is separable over Z' by [Rev]. By transitivity of separability [L V–V, II.5.1.2], B is separable over Z'. Then, by [De–In, II.3.8], so is Z.

Consider now assertion (b). First, denote by F and F' the fraction fields of Z and Z'. One has $F' = F^G$ and, since G acts faithfully on Z, then $\dim_{F'} F = |G|$. As is well known, it follows from Nakayama's lemma that, for every maximal ideal m' of Z', one has

$$\dim_{Z'/m'} Z/m'Z \geqslant \dim_{F'} F = |G|. \qquad (**)$$

Now, let m be a maximal ideal of Z of codimension 1, and $m' = m \cap Z$. Then the maximal ideals of Z containing $m'Z$ are precisely the G-conjugates of m, say $m = m_1, ..., m_t$, where $t = |Gm|$; they have codimension 1 as well. On the other hand, since Z is separable over Z', then $Z/m'Z$ is separable over $Z'/m' = k$ by [De–In, II.1.7]. Hence, by [*loc. cit.*, II.2.4], the finite dimensional k-algebra $Z/m'Z$ has zero radical. It follows that $Z/m'Z$ is the direct product of the Z/m_i, where $1 \leqslant i \leqslant t$; hence has dimension t. Together with $(**)$, this gives $t = |G|$. The proposition is proved.

2.9. *Completion of the proof.* We can now conclude as follows. It is well known and easy to see that Z is a polynomial algebra over k in $2n$ variables; therefore the set of maximal ideals of Z of codimension 1 is in bijection with the affine space k^{2n}. Proposition 2.8(b) then implies that $|G|$ divides $|k^{2n}|$, which is a power of l. Since, on the other hand, our construction was made so that l is prime to $|G|$, this gives $|G| = 1$. The proof of Theorem 2 is complete.

2.10. *Linear actions.* Recall the notations and hypotheses of Theorem 2; in particular, K denotes a field of characteristic zero. For the sake of completeness, we shall give a shorter proof, under the additional hypothesis that the action of G is linearizable; which means that some conjugate of G in $\operatorname{Aut}_K A_n(K)$ preserves the natural filtration of $A_n(K)$. Namely, let us prove the following.

PROPOSITION. *Let G be a finite group of K-automorphisms of $A_n(K)$; assume that the action of G is linearizable, then $K_0(A_n(K)^G) \simeq \mathbf{Z}^{\operatorname{irr}_K(G)}$, where $\operatorname{irr}_K(G)$ denotes the number of irreducible representations of G over K. In particular, if $A_n(K)^G$ is K-isomorphic to $A_n(K)$, then G is trivial.*

Proof. Set $A_n(K) = A$. Firstly, note that $K_0(A) \simeq \mathbf{Z}$ by Quillen's theorem [Qui, Section 6, Theorem 7]. Second, since conjugate subgroups have isomorphic algebras of invariants, we may assume that G itself preserves the canonical filtration of A. Then, as observed in [A-H-V, Theorem 2.1], it also follows from Quillen's theorem that $K_0(A * G) \simeq K_0(KG)$, and the latter is isomorphic to $\mathbf{Z}^{\mathrm{irr}_K(G)}$. On the other hand, since the only invertible elements of A are the nonzero scalars, which are central, then A has no nontrivial inner automorphism. Therefore, it follows from [Mo 1, 2.5–2.6] that $A * G$ and A^G are Morita equivalent; thus $K_0(A^G) \simeq K_0(A * G) \simeq \mathbf{Z}^{\mathrm{irr}_K(G)}$.

Assume now that $A^G \simeq A$. Since $K_0(A) \simeq \mathbf{Z}$, it follows $\mathrm{irr}_K(G) = 1$; hence G has only one irreducible representation over K, the trivial one. But, since KG is semisimple, the sum of all its irreducible representations is faithful; it follows that G is trivial.

References

[A-H-V] J. Alev, T. J. Hodges, and J.-D. Velez, Fixed rings of the Weyl algebra $A_1(\mathbf{C})$, *J. Algebra* **130** (1990), 83–96.

[Bou] N. Bourbaki, "Groupes et algèbres de Lie," Chaps. IV–VI, VII–VIII, Hermann, Paris, 1968, 1975.

[De–In] F. DeMeyer and E. Ingraham, "Separable Algebras over Commutative Rings," Lect. Notes in Math., Vol. 181, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[Di–Fo] W. Dicks and E. Formanek, Poincaré series and a problem of S. Montgomery, *Linear Multilinear Algebra* **12** (1982), 21–30.

[Dix] J. Dixmier, "Algèbres Enveloppantes," Gauthier-Villars, Paris/Bruxelles/Montréal, 1974.

[Duf] M. Duflo, Idéaux primitifs dans les algèbres enveloppantes, *Ann. of Math.* **105** (1977), 107–120.

[Jan] J. C. Jantzen, "Einhüllende Algebren halbeinfacher Lie-Algebren," Springer-Verlag, Berlin/Heidelberg/New York, 1983.

[Kar] V. K. Karchenko, Algebras of invariants of free algebras, *Algebra and Logic* **17** (1978), 316–321.

[Kr–Sm] H. Kraft and L. Small, Invariant algebras and completely reducible representations, *Math. Research Letters* **1** (1994), 297–307.

[Kr–Le] G. Krause and T. Lenagan, "Growth of Algebras and Gelfand-Kirillov Dimension," Res. Notes in Math., Vol. 116, Pitman, Boston/London, 1985.

[Mo 1] S. Montgomery, "Fixed Rings of Finite Automorphisms Groups of Associative Rings," Lect. Notes in Math., Vol. 818, Springer-Verlag, Berlin/Heidelberg/New York, 1980.

[Mo 2] S. Montgomery, X-inner automorphisms of filtered algebras, *Proc. Amer. Math. Soc.* **83** (1981), 263–268.

[Mo 3] S. Montgomery, Prime ideals in fixed rings, *Comm. Algebra* **9** (1981), 423–449.

[Mo–Gu] S. Montgomery and R. M. Guralnick, On invertible bimodules and automorphisms of noncommutative rings, *Trans. Amer. Math. Soc.*

[Mo–Sm] S. Montgomery and L. Small, Fixed rings of noetherian rings, *Bull. London Math. Soc.* **13** (1981), 33–38.

[L V V] L. Le Bruyn, M. Van den Bergh, and F. Van Oystaeyen, "Graded Orders," Birkhäuser, Boston/Basel, 1988.

[Qui] D. Quillen, Higher algebraic K-thery, *in* "Algebraic K-Theory" (H. Bass, Ed.), Lect. Notes in Math., Vol. 341, Springer-Verlag, Berlin/Heidelberg/New York, 1973.

[Rev] P. Revoy, Algèbres de Weyl en caractéristique *p*, *C. R. Acad. Sci. Paris* **276** (1973), 225–228.

[Smi] P. Smith, Can the Weyl algebra be a fixed ring?, *Proc. Amer. Math. Soc.* **107** (1989), 587–589.

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The Enumeration of Totally Symmetric Plane Partitions

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A plane partition is totally symmetric if, as an order ideal of $\mathbf{N}^3$, it is invariant under all (six) permutations of the coordinate axes. We prove an explicit product formula for the number of totally symmetric plane partitions that fit in a cube of order n. This settles a conjecture published in 1980 by Andrews (who also attributed it to Macdonald and Stanley) and finishes the program of enumerating plane partitions belonging to each of the 10 symmetry classes proposed by Stanley in 1986. As a corollary of the proof, we also obtain a new proof of the formula, first obtained by Mills, Robbins, and Rumsey, for the number of plane partitions that are invariant under cyclic permutations and transposed complementation.

0. INTRODUCTION

A plane partition π is a rectangular array of nonnegative integers with decreasing rows and columns and finitely many nonzero entries. For our purposes, it will be preferable to regard a plane partition as an order ideal of $\mathbf{N}^3$, i.e., a finite subset of $\mathbf{N}^3$ such that $(i, j, k) \in \pi$ implies $(i', j', k') \in \pi$ whenever $i \geqslant i'$, $j \geqslant j'$, and $k \geqslant k'$. (One recovers the array from the order ideal by setting $\pi_{ij} = |\{k : (i, j, k) \in \pi\}|$.)

The study of plane partitions was initiated by MacMahon in the late 19th century [6]. MacMahon proved, among other things, a simple product formula for the generating function for plane partitions π such that $\pi \subset B(a, b, c)$, where

$$B(a, b, c) := \{(i, j, k) \in \mathbf{N}^3 : i < a, j < b, k < c\}$$

denotes an array of lattice points of size $a \times b \times c$.

By permuting the coordinate axes, one obtains an action of S_3 on the set of plane partitions. Thus for each subgroup G of S_3, it is natural to attempt the enumeration of various classes of G-invariant plane partitions. (This idea can also be traced back to MacMahon; see Sections 9–10 of [6].)

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Let $N_G(B, q)$ denote the generating function for G-invariant plane partitions π such that $\pi \subset B = B(a, b, c)$; i.e.,

$$N_G(B, q) = \sum_{\pi \subset B} q^{|\pi|}. \tag{0.1}$$

Without loss of generality, we may assume that B is G-invariant. By the result of MacMahon one has an explicit formula for $N_G(B, q)$ in the case of the trivial group. MacMahon also conjectured a similar formula in the case $G = S_2$; this conjecture was later proved independently by Andrews, Gordon, and Macdonald (for references and further background, see [10]). In 1977, Macdonald conjectured a formula for the case $G = C_3$ that was later proved by Mills *et al.* in 1982 [7]. Macdonald also noted that the formulas in the three cases could be given a uniform description (although at the time, the C_3 case was merely a conjecture). To explain Macdonald's formula, let B/G denote the set of G-orbits of B, and let $r(i, j, k) = i + j + k + 1$. Note that r is S_3-invariant, so it is well-defined on B/G.

THEOREM 0.1. *For $G \neq S_3$, we have*

$$N_G(B, q) = \prod_{x \in B/G} \frac{1 - q^{|x|(r(x)+1)}}{1 - q^{|x| r(x)}}.$$

This result cannot be extended to $G = S_3$ since the above product is not even a polynomial in that case. Nevertheless, it was conjectured shortly thereafter by several people (including one or more of Andrews, Macdonald, and Stanley) that the result remains true for $G = S_3$ in the special case $q = 1$ [1]. The main contribution of the present paper is a proof of this conjecture (see Corollary 5.2). By combining this with the results for the other cases, we thus obtain the following unified formula for $N_G(B) = N_G(B, 1)$, the number of G-invariant plane partitions.

THEOREM 0.2. *If G is a subgroup of S_3, then*

$$N_G(B) = \prod_{x \in B/G} \frac{r(x) + 1}{r(x)}.$$

We should remark that Andrews and Robbins have independently conjectured that if one modifies the generating function (0.1), replacing the number of points $|\pi|$ with the number of orbits of points $|\pi/G|$, then there is a simple product formula in the case $G = S_3$. In fact, if we define

$$N'_G(B, q) = \sum_{\pi \subset B} q^{|\pi/G|},$$

then the conjecture of Andrews and Robbins (as formulated by Stanley in [10]) can be stated as follows.

Conjecture 0.3 If $G = S_3$, then

$$N'_G(B, q) = \prod_{x \in B/G} \frac{1 - q^{r(x)+1}}{1 - q^{r(x)}}.$$

This conjecture is still unsolved.

Let $\pi \mapsto \pi^t$ and $\pi \mapsto \pi^r$ denote the symmetries corresponding to transposition and rotation (i.e., $(i, j, k)^t = (j, i, k)$ and $(i, j, k)^r = (j, k, i)$). For the set of plane partitions contained in a given box $B = B(a, b, c)$, there is an additional symmetry

$$\pi \mapsto \pi^c = \{(i, j, k) \in B: (a - i - 1, b - j - 1, c - k - 1) \notin \pi\},$$

knows as the complement. This symmetry, together with the previous symmetries, generates a group $\Gamma = \langle t, r, c \rangle$ isomorphic to the dihedral group of order 12. In 1985, Mills, Robbins, and Rumsey conjectured an explicit product formula for $N_\Gamma(B)$ [8]. This suggested to Stanley the idea of studying $N_G(B)$ for all subgroups G of Γ [10]. This leads to 10 essentially distinct enumerative problems, and as of 1986 when Stanley's paper was published, explicit product formulas had been proved for 7 of the 10 cases and were conjectured for the remaining three cases, namely

1. $G = \langle t, r \rangle$ (totally symmetric),
2. $G = \langle r, c \rangle$ (cyclically symmetric and self-complementary),
3. $G = \Gamma = \langle t, r, c \rangle$ (totally symmetric and self-complementary).

Recently, solutions have been found for Case 2 (by Kuperberg [5]) and for Case 3 (by Andrews [3]). Thus with the solution of Case 1 in the present paper, the number of G-invariant plane partitions $N_G(B)$ has been proved to have a simple product formula for every subgroup G of Γ.

Outline of the Proof. It will be convenient for what follows to let TS(n) denote the number of totally symmetric plane partitions π such that $\pi \subset B(n, n, n)$. Similarly, let CS(n) denote the analogous number of cyclically symmetric plane partitions. The proof has four steps:

1. Interpret TS(n) and CS(n) as the number of configurations of nonintersecting paths in $\mathbf{Z}^2$ subject to certain boundary conditions. (Similar interpretations can be given for any of the 10 symmetry classes.)

2. Convert the interpretation for TS(n) into a Pfaffian of order (roughly) n. Similarly convert the interpretation for CS(n) into a determinant of order n.

3. Reduce the Pfaffian for $\mathrm{TS}(n)$ to a determinant of order (roughly) $n/2$.

4. Embed the determinants for $\mathrm{TS}(n)$ and $\mathrm{TS}(n-1)$ into a single determinant of order n, and recognize the result as essentially the same as the determinant for $\mathrm{CS}(n)$ in Step 2, thereby deducing the relationship

$$2^{n-1}\mathrm{CS}(n) = \mathrm{TS}(n)\,\mathrm{TS}(n-1). \tag{0.2}$$

By Theorem 0.1, one knows already that there is a product formula for $\mathrm{CS}(n)$ (a result first proved by Andrews in [2]), so by induction it follows that there is a product formula for $\mathrm{TS}(n)$. This completes the proof.

A singular (perhaps crucial) feature of the methods we employ in Steps 3 and 4 is that rather than working directly with explicit matrices, we instead work with the underlying bilinear forms and avoid the complications that might be imposed by any particular choice of basis. Indeed, a benefit of this approach is that after the reductions have been carried out, it is easy to spot a new basis for the bilinear form attached to $\mathrm{TS}(2n-1)$ that yields a matrix whose determinant is known to be the number of plane partitions belonging to the class invariant under $G = \langle r, tc \rangle$. More specifically (see Theorem 5.3), we deduce that

$$\mathrm{TS}(2n-1) = 2^{2n-1}\mathrm{CSTC}(2n), \tag{0.3}$$

where $\mathrm{CSTC}(n)$ denotes the number of cyclically symmetric, transpose-complementary (i.e., of $\langle r, tc \rangle$-invariant) plane partitions π such that $\pi \subset B(n, n, n)$. (We should remark that $\mathrm{CSTC}(n)$ is nonzero only for even n). Thus, given the product formula for $\mathrm{TS}(n)$, we now have another proof of the product formula for $\mathrm{CSTC}(2n)$, a result first obtained by Mills *et al.* [9]. It is interesting to note that their proof has a structure similar to ours, in that both proofs ultimately rely on previous solutions of the cyclically symmetric case.

It would be interesting to give bijective explanations of (0.2) or (0.3).

In view of Conjecture 0.3, it would be natural to try to find a q-analogue of the proof of (0.2). It is straightforward to give q-analogues of Steps 1 and 2, but difficulties arise in Step 3. The basic obstruction is the linear transformation $f \mapsto f^*$ defined in Section 3. Although it has a simple q-analogue, it turns out not to be an involution for general q, a property that is crucial in the following.

1. Nonintersecting Paths

Define a graph on $\mathbf{Z}^2$, the set of lattice points in the plane, with edges between nearest neighbors (i.e., $u, v \in \mathbf{Z}^2$ are adjacent if and only if

$u-v=(\pm 1, 0)$ or $(0, \pm 1)$). Orient the edges north and west, so that an edge is directed from u to v if $v-u=(0, 1)$ or $(-1, 0)$. Throughout this paper, a *path* shall be understood to mean a directed path in this graph; i.e., a vertex sequence $(v_0, v_1, ..., v_l)$ in which $v_{i+1}-v_i=(0, 1)$ or $(-1, 0)$ for all i. A collection of paths will be said to be *nonintersecting* if the vertex sets of the paths are disjoint. The transpose of a path $P=(v_0, ..., v_l)$ is defined to be the path $P'=(v_l', ..., v_0')$ where $(i,j)'=(j, i)$.

The paths from $(i, 0)$ to $(0, j)$ are in one-to-one correspondence with the Ferrers diagrams of ordinary partitions with $j+1$ rows and $i+1$ columns. To be specific about the correspondence, let us define the Ferrers diagram associated to the path $P=(v_0, ..., v_l)$ to be the order ideal of $\mathbf{N}^2$ generated by $v_0, ..., v_l$). For an example, see Fig. 1.

Let $\mathscr{P}$ denote the set of paths P with the property that the initial and terminal points of P are of the form $(j, 0)$ and $(0, j)$ for some $j \geqslant 0$. Set $X_n=\{(j, 0): 0 \leqslant j<n\}$.

LEMMA 1.1. *There is a one-to-one correspondence between cyclically symmetric plane partitions and non-intersecting (finite) subsets of $\mathscr{P}$. Furthermore, this correspondence can be chosen so that if $\pi \leftrightarrow \{P_1, P_2, ...\}$, then*

(a) $\pi \subset B(n, n, n)$ *if and only if the initial point of each path P_i belongs to X_n.*

(b) $\pi' \leftrightarrow \{P_1', P_2', ...\}$.

Proof. Let $\pi \subset \mathbf{N}^3$ be a plane partition, and define π_s: $\{(i,j,k) \in \pi$: $\min(i,j,k)=s\}$. Let r be the largest integer such that $\pi_r \neq \varnothing$. The mapping $\pi \mapsto \{\pi_0, ..., \pi_r\}$ is injective and commutes with the action of S_3 on the coordinates. To describe the range of this map, consider the set Π_1 consisting of all plane partitions σ having "unit thickness" (i.e., $(1, 1, 1) \notin \sigma$). Note that $\pi_0 \in \Pi_1$, and more generally, $\bar{\pi}_i$: $=-(i, i, i)+\pi_i \in \Pi_1$. If we are given an arbitrary sequence $(\sigma_0, ..., \sigma_r)$ with $\sigma_i \in \Pi_1$, then there will exist a plane partition π such that $\bar{\pi}_i=\sigma_i$ if and only if the shells σ_i "fit together"; i.e.,

$$(i,j,0) \in \sigma_{s+1} \Rightarrow (i+1, j+1, 0) \in \sigma_s \tag{1.1a}$$

$$(i, 0, j) \in \sigma_{s+1} \Rightarrow (i+1, 0, j+1) \in \sigma_s \tag{1.1b}$$

$$(0, i, j) \in \sigma_{s+1} \Rightarrow (0, i+1, j+1) \in \sigma_s \tag{1.1c}$$

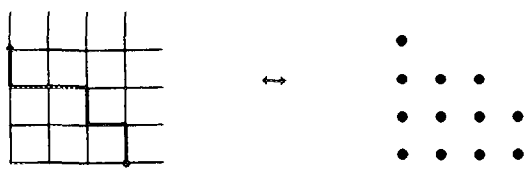

FIGURE 1

for all $i, j, s \geqslant 0$. For cyclically symmetric plane partitions, there is redundancy in these conditions; it suffices merely to impose (1.1a), say.

For $\sigma \in \Pi_1$, let us define $F(\sigma) = \{(i, j): (i, j, 0) \in \sigma\}$. Since $F(\sigma)$ is the Ferrers diagram of an ordinary partition, there is a path $P(\sigma)$ associated to $F(\sigma)$, as we defined earlier. Condition (1.1a) is equivalent ot the condition that $F(\sigma_{s+1})$ is a subset of $F(\sigma_s)$ that contains no elements of the outer boundary of $F(\sigma_s)$, or equivalently, that $P(\sigma_{s+1})$ lies strictly to the southwest of $P(\sigma_s)$. (That is, $P(\sigma_{s+1})$ and $P(\sigma_s)$ do not intersect.)

If $\sigma \in \Pi_1$ is C_3-invariant, then the mapping $\sigma \mapsto P(\sigma)$ is injective with range $\mathscr{P}$. Hence, the mapping $\pi \mapsto \{P(\bar{\pi}_0), ..., P(\bar{\pi}_r)\}$ defines a bijection between the set of cyclically symmetric plane partitions and the collection of non-intersecting subsets of $\mathscr{P}$.

To complete the proof, note that the bijection we have just defined commutes with the transpose operation, and (assuming cyclic symmetry) π will fit in $B(n, n, n)$ if and only if the initial point of $P(\bar{\pi}_0)$ (the largest shell) is of the form $(i, 0)$ with $i < n$. ∎

Let $\mathscr{Q}$ denote the set of paths whose initial and terminal points are of the form $(i, 0)$ and (j, j) for arbitrary $i \geqslant j \geqslant 0$.

Lemma 1.2. *There is a one-to-one correspondence between totally symmetric plane partitions π and non-intersecting (finite) subsets of $\mathscr{Q}$. Furthermore, this correspondence can be chosen so that $\pi \subset B(n, n, n)$ if and only if the initial points of the corresponding set of paths belong to X_n.*

Proof. Given a path $P \in \mathscr{Q}$, one can create a path $P^* \in \mathscr{P}$ by adjoining P^t. The paths so created are the transpose-invariant paths in $\mathscr{P}$. By the previous lemma, sets of non-intersecting paths of this type are in one-to-one correspondence with cyclically symmetric plane partitions π such that $\pi = \pi^t$. ∎

The following observation is due to Mills, Robins, and Rumsey.

Lemma 1.3. *There is a one-to-one correspondence between $\langle r, tc \rangle$-invariant plane partitions contained in $B(2n, 2n, 2n)$, and n-tuples of nonintersecting paths $(P_0, ..., P_{n-1})$ in which the initial and terminal points of P_i are $(2i, i)$ and $(i, 2i)$, respectively.*

Proof. See Theorem 1 and p. 48 of [9]. ∎

2. Determinants and Pfaffians

By using the basic theorems on enumeration of non-intersecting paths, we can easily convert the interpretations of the previous section into determinants and Pfaffians.

For $j \geqslant 0$, define a_j to be the total number of paths in $\mathscr{Q}$ with initial point $(j, 0)$. For $j \geqslant i \geqslant 0$, define a_{ij} to be the number of pairs of non-intersecting paths $P, Q \in \mathscr{Q}$ such that the initial point of P is $(i, 0)$ and the initial point of $\mathscr{Q}$ is $(j, 0)$.

LEMMA 2.1. *We have*

(a) $a_j = 2^j$.

(b) $a_{ij} = \sum_{i \leqslant r < j} \binom{i+j}{r}$.

Proof. The first assertion is obvious. For the second, note that the total number of pairs of paths P, Q, with or without intersections, in which P is a path from $(i, 0)$ to (k, k) and Q is a path from $(j, 0)$ to (l, l) is $\binom{i}{k}\binom{j}{l}$. Hence by the fundamental theorem on non-intersecting paths (Theorem 1 of [4] or Theorem 1.2 of [11]), it follows that

$$a_{ij} = \sum_{0 \leqslant k < l \leqslant j} \binom{i}{k}\binom{j}{l} - \sum_{0 \leqslant k < l \leqslant i} \binom{i}{l}\binom{j}{k}.$$

Applying the change of variables $l \to j - l$ to the first sum and $l \to i - l$ to the second sum, we obtain

$$a_{ij} = \sum_{i \leqslant k + l < j} \binom{i}{k}\binom{j}{l}.$$

Use Vandermonde's summation to simplify the result. ∎

LEMMA 2.2. *We have* $\mathrm{TS}(2n+1) = \mathrm{Pf}[a_{ij}^*]_{-1 \leqslant i, j \leqslant 2n}$, *where* a_{ij}^* *is the unique skew-symmetric matrix satisfying*

$$a_{ij}^* = \begin{cases} (-1)^{i+j-1} + a_j & \text{if } \ -1 = i < j, \\ (-1)^{i+j-1} + a_{ij} & \text{if } \ 0 \leqslant i < j. \end{cases}$$

Proof. By Theorem 4.1(b) of [11], together with Lemma 2.1 above, it follows that $\mathrm{Pf}[a_{ij}^*]_{-1 \leqslant i, j < 2n+1}$ is the number of sequences of non-intersecting paths $(P_1, P_2, \ldots)$ in $\mathscr{Q}$ in which the initial points of the paths are of the form $(i_1, 0), (i_2, 0), \ldots$, where $0 \leqslant i_1 < i_2 < \ldots \leqslant 2n$. (It should be noted that the rows and columns of the matrix in [11] are a permutation of the ones we have used here, but it is easy to check that the particular rearrangement we have chosen does not alter the sign of the Pfaffian.) Apply Lemma 1.2. ∎

LEMMA 2.3. *We have* $\mathrm{TS}(2n)=\mathrm{Pf}[a_{ij}^{**}]_{-2\leqslant i,j<2n}$, *where* a_{ij}^{**} *is the unique skew-symmetric matrix satisfying*

$$a_{ij}^{**}=\begin{cases}(-1)^{i+j-1} & \text{if} \quad -2=i<j,\\ (-1)^{i+j-1}+a_j & \text{if} \quad -1=i<j,\\ (-1)^{i+j-1}+a_{ij} & \text{if} \quad 0\leqslant i<j.\end{cases}$$

Proof. Proceed as in the proof of Lemma 2.2, using Theorem 4.1(c) of [11]. ∎

The following result was first proved by Andrews (cf. [2, Theorem 4]).

LEMMA 2.4. $\mathrm{CS}(n)=\det\left[\delta_{ij}+\binom{i+j}{i}\right]_{0\leqslant i,j<n}$.

Proof. Let $B=\left[\binom{i+j}{i}\right]_{0\leqslant i,j<n}$. The number of paths from $(i,0)$ to $(0,j)$ is $\binom{i+j}{i}$. By the basic theorem of Gessel and Viennot on the enumeration of non-intersecting paths (e.g., see Theorem 1.2 of [11]), it follows that the principal minor of B obtained by selecting the rows and columns indexed by $\{i_1<i_2<\ldots<i_r\}$ is the number of r-tuples of nonintersecting paths $(P_1,\ldots,P_r)$ in which the initial point of P_k is $(i_k,0)$, and the terminal point of P_k is $(0,i_k)$. By Lemma 1.1, it follows that $\mathrm{CS}(n)$ is the sum of all principal minors of B, i.e., $\det(I+B)$. ∎

The following result (as well as the proof we give below) is the special case $x=1$ of Theorem 3 in [9].

LEMMA 2.5. $\mathrm{CSTC}(2n)=\det\left[\binom{i+j}{2j-i}\right]_{0\leqslant i,j<n}$.

Proof. The number of paths from $(2i,i)$ to $(j,2j)$ is $\binom{i+j}{2j-i}$. Hence by the Gessel Viennot theorem, $\det\left[\binom{i+j}{2j-i}\right]_{0\leqslant i,j<n}$ is the number of n-tuples of nonintersecting paths $(P_0,\ldots,P_{n-1})$ in which the initial and terminal points of P_i are $(2i,i)$ and $(i,2i)$. Apply Lemma 1.3. ∎

3. LINEAR ALGEBRA

For a Laurent polynomial $f\in\mathbf{Q}[t,t^{-1}]$, let $[t^i]f(t)$ denote the coefficient of t^i in $f(t)$. We may define a skew-symmetric $\mathbf{Q}$-bilinear form on $\mathbf{Q}[t]$ by setting

$$(f,g):=[1]\,\frac{t}{1+t}\,(f(t)g(1/t)-f(1/t)g(t))$$

for all $f,g\in\mathbf{Q}[t]$. Note that the numerator of the above expression does vanish at $t=-1$, so the quotient is a well-defined Laurent polynomial.

LEMMA 3.1. *For all $j > i \geqslant 0$ and $f \in \mathbf{Q}[t]$, we have*

(a) $(t^i, t^j) = (-1)^{i+j-1}$.

(b) $((1-t)^i, (1-t)^j) = -a_{ij} = -\sum_{i \leqslant r < j} \binom{i+j}{r}$.

(c) $(f, 1) = f(-1) - f(0)$.

(d) $(f, t) = 2f(0) - f'(0) - f(-1)$.

Proof. These are all routine verifications. ∎

Let $V_k = t^{-k}\mathbf{Q}[t]$ denote the vector space spanned by Laurent monomials of degree at least $-k$. In the following, we will show that the matrices of Lemmas 2.2 and 2.3 can be viewed as the matrices of certain simple bilinear forms on V_1 and V_2, respectively.

Let $f \mapsto f^*$ denote the involution on $\mathbf{Q}[t]$ defined by setting $f^*(t) := f(1-t)$. There is a unique skew-symmetric $\mathbf{Q}$-bilinear form $\langle \cdot, \cdot \rangle$ on V_1 such that

$$\langle f, g \rangle = (f, g) - (f^*, g^*), \tag{3.1}$$

$$\langle t^{-1}, f \rangle = f(-1) + f(2) = (f + f^*)(-1) \tag{3.2}$$

for all $f, g \in \mathbf{Q}[t]$. By Lemma 3.1, note that for $j > i \geqslant 0$ we have

$$\langle t^i, t^j \rangle = (-1)^{i+j-1} + a_{ij},$$

$$\langle t^{-1}, t^i \rangle = (-1)^i + 2^i = (-1)^i + a_i.$$

In other words, $[\langle t^i, t^j \rangle]_{-1 \leqslant i, j \leqslant 2n}$ is the matrix of Lemma 2.2, and so we obtain

$$\mathrm{TS}(2n+1) = \mathrm{Pf}[\langle t^i, t^j \rangle]_{-1 \leqslant i, j \leqslant 2n}.$$

More generally, since $\mathrm{Pf}(XAX') = \det(X)\,\mathrm{Pf}(A)$ (e.g., [11, Sect. 2]) it follows that

$$\mathrm{TS}(2n+1) = \mathrm{Pf}[\langle u_i, u_j \rangle]_{-1 \leqslant i, j \leqslant 2n}, \tag{3.3}$$

for any sequence $u_{-1}, u_0, u_1, \ldots \in V_1$ with u_i monic and $\deg(u_i) = i$.

Let $\mathbf{Q}[t]^{\pm} = \{ f \in \mathbf{Q}[t] : f(1-t) = \pm f(t) \}$ denote the eigenspaces of $*$ on $\mathbf{Q}[t]$. It is easy to check that $\mathbf{Q}[t]^+ = \mathbf{Q}[t^2 - t]$, and $\mathbf{Q}[t]^- = (t - 1/2)\,\mathbf{Q}[t^2 - t]$. Now extend the definition of $*$ to V_1 by setting $(1/t)^* = -1/t$, and let $V_1^{\pm}$ denote the corresponding eigenspaces. Note that $V_1^+ = \mathbf{Q}[t]^+$ and $V_1^- = \mathbf{Q}t^{-1} \oplus \mathbf{Q}[t]^-$. By (3.1) and (3.2) it follows that $\langle f^*, g^* \rangle = -\langle f, g \rangle$ for all $f, g \in V_1$, so $\langle f, g \rangle = 0$ for $f, g \in V_1^+$ or $f, g \in V_1^-$.

If $B = [b_{ij}]_{1 \le i,j \le 2n}$ is any skew-symmetric matrix of even order such that $b_{2i,2j} = 0$, then it can be easily shown that

$$\mathrm{Pf}(B) = \det[b_{2i-1,2j}]_{1 \le i,j \le n}. \tag{3.4}$$

For example, this follows directly from the definition of the Pfaffian in [11,Section 2].

Therefore, if $u_0^+, u_1^+, u_2^+, \ldots$ (resp., $u_0^-, u_1^-, u_2^-, \ldots$) is a monic basis for V_1^+ (resp., V_1^-), chosen so that $\deg(u_i^+) = 2i$ (resp., $\deg(u_i^-) = 2i-1$), then the sequence $u_0^-, u_0^+, u_1^-, u_1^+, \ldots$ meets the constraints of (3.3), and so by (3.4) we obtain

$$\mathrm{TS}(2n+1) = \det[\langle u_i^-, u_j^+ \rangle]_{0 \le i,j < n}. \tag{3.5}$$

It will be useful to note here that the bilinear form $\langle \cdot, \cdot \rangle$ can be simplified somewhat when restricted to $V_1^- \otimes V_1^+$. Indeed, by (3.1) and (3.2), we have

$$\langle f, g \rangle = 2(f, g), \qquad \langle t^{-1}, g \rangle = 2g(-1) \tag{3.6}$$

for all $f \in \mathbf{Q}[t]^-$ and $g \in \mathbf{Q}[t]^+$.

For the analysis of $\mathrm{TS}(2n)$, extend the definition of $\langle \cdot, \cdot \rangle$ from V_1 to V_2 by defining

$$\langle t^{-2}, at^{-1} + f(t) \rangle = a - f(-1)$$

for all $f \in \mathbf{Q}[t]$ and $a \in \mathbf{Q}$. By Lemma 3.1, it is easy to see that $[\langle t^i, t^j \rangle]_{-2 \le i,j < n}$ is the matrix of Lemma 2.3, so we have

$$\mathrm{TS}(2n) = \mathrm{Pf}[\langle t^i, t^j \rangle]_{-2 \le i,j < 2n}. \tag{3.7}$$

It will be convenient for what follows to eliminate the first two rows and columns of this Pfaffian. To achieve this, define a linear transformation $\psi\colon V_2 \to V_2$ by setting

$$\psi(at^{-2} + bt^{-1} + f(t)) = at^{-2} + (b + f(-1))t^{-1} + f(t)$$

for all $f \in \mathbf{Q}[t]$ and $a, b \in \mathbf{Q}$. The matrix of ψ with respect to the ordered basis $t^{-2}, t^{-1}, t^0, \ldots$, is obviously upper triangular with a unit diagonal, so replacing $\langle t^i, t^j \rangle$ with $\langle \psi(t^i), \psi(t^j) \rangle$ will leave the Pfaffian unchanged. However, since $\langle \psi(t^{-2}), \psi(t^{-1}) \rangle = 1$ and $\langle \psi(t^{-2}), \psi(f) \rangle = 0$ for all $f \in \mathbf{Q}[t]$, it follows by a Laplace-type expansion that

$$\mathrm{Pf}[\langle t^i, t^j \rangle]_{-2 \le i,j < 2n} = \mathrm{Pf}[\langle \psi(t^i), \psi(t^j) \rangle]_{0 \le i,j < 2n}. \tag{3.8}$$

This can be formulated in a slightly more elegant way by defining

$$\langle f, g\rangle' := \langle f+f(-1)\,t^{-1}, g+g(-1)\,t^{-1}\rangle = \langle f, g\rangle + (fg^* - f^*g)(-1)$$

for all $f, g \in \mathbf{Q}[t]$. By (3.7) and (3.8), it follows that

$$TS(2n) = \mathrm{Pf}[\langle u_i, u_j\rangle']_{0 \leqslant i, j < 2n}, \tag{3.9}$$

for any sequence $u_0, u_1, \ldots \in \mathbf{Q}[t]$ with u_i monic and $\deg(u_i) = i$.

In view of (3.1), it is easy to see that $\langle f^*, g^*\rangle' = -\langle f, g\rangle'$ for all $f, g \in \mathbf{Q}[t]$, so again we have $\langle f, g\rangle' = 0$ if $f, g \in \mathbf{Q}[t]^+$ or $f, g \in \mathbf{Q}[t]^-$. Thus if $u_0^\pm, u_1^\pm, u_2^\pm, \ldots$ are monic bases for $\mathbf{Q}[t]^\pm$ with $\deg(u_i^-) = 2i+1$ and $\deg(u_i^+) = 2i$, then the sequence $u_0^+, u_0^-, u_1^+, u_1^-, \ldots$ will meet the constraints of (3.9), and so by (3.4) we obtain

$$\mathrm{TS}(2n) = \det[\langle u_i^+, u_j^-\rangle']_{0 \leqslant i, j < n}. \tag{3.10}$$

Note that $\langle f, g\rangle'$ simplifies upon restriction to $V_1^+ \otimes V_1^-$. Indeed, by (3.1), we have

$$\langle f, g\rangle' = 2(f, g) - 2f(-1)g(-1) \tag{3.11}$$

for all $f \in \mathbf{Q}[t]^+$ and $g \in \mathbf{Q}[t]^-$.

4. More Linear Algebra

In this section, we will show by means of suitable linear transformations that the forms occurring in (3.5) and (3.10) can be replaced with certain *symmetric* bilinear forms on $\mathbf{Q}[t]^+$ and $\mathbf{Q}[t]^-$, respectively.

For the first of these, let $\varepsilon: \mathbf{Q}[t] \to \mathbf{Q}$ be some given linear functional, and define a pair of linear transformations $\phi_1: V_1^+ \to V_1^-$ (depending on ε) and $\phi_2: V_1^+ \to V_1^+$ by setting

$$\phi_1(f)(t) = \frac{f(t) - f(1/2)}{t - 1/2} + \varepsilon(f)\,t^{-1}$$

$$\phi_2(f)(t) = \frac{(t+1)(t-2)}{t(t-1)}(f(t) - f(0)) + f(0),$$

for all $f \in \mathbf{Q}[t]^+ = V_1^+$. Note that for $f \in \mathbf{Q}[t]^+$, $f(0) = 0$ implies $f(1) = 0$ and hence $t(t-1) \mid f(t)$ in such cases. (Thus ϕ_2 is well-defined.) It is easy to see that $\phi_2(f)$ preserves the degree and leading coefficient of f. Therefore, if u_i^+ $(i = 0, 1, 2, \ldots)$ is a monic basis of V_1^+ with $\deg(u_i^+) = 2i$, then the matrix of ϕ_2 with respect to this basis is upper triangular with a unit diagonal. Similarly, if u_i^- $(i = 0, 1, 2, \ldots)$ is a monic basis of V_1^- with

$\deg(u_i) = 2i-1$, then the matrix of ϕ_1 with respect to this pair of bases is also upper triangular with unit diagonal, except for the fact that the $(0, 0)$-entry is $\varepsilon(1)$. Thus by (3.5), it follows that

$$\varepsilon(1) \cdot \mathrm{TS}(2n+1) = \det[\langle \phi_1(u_i^+), \phi_2(u_j^+)\rangle]_{0 \leq i, j \leq n}.$$

LEMMA 4.1. *We can choose ε so that $\varepsilon(1) = 2$ and for all $f, g \in \mathbf{Q}[t]^+$,*

$$\langle \phi_1(f), \phi_2(g)\rangle = 4[1]\, f(t)\, g(1/t).$$

COROLLARY 4.2. *If $u_0, u_1, u_2, \ldots, \in \mathbf{Q}[t]^+$ are monic and $\deg(u_i) = 2i$, then*

$$\mathrm{TS}(2n+1) = 2^{2n+1} \det[b_{ij}]_{0 \leq i, j \leq n},$$

where $b_{ij} = [1]\, u_i(t)\, u_j(t^{-1})$.

Proof of Lemma 4.1. First suppose $g(t) = 1$, so that $\phi_2(g) = 1$. By (3.6), we have

$$\begin{aligned}\langle \phi_1(f), \phi_2(g)\rangle &= \left\langle \frac{f(t) - f(1/2)}{t - 1/2} + \varepsilon(f)\, t^{-1}, 1 \right\rangle \\ &= 2\left\langle \frac{f(t) - f(1/2)}{t - 1/2}, 1 \right\rangle + 2\varepsilon(f),\end{aligned}$$

and $4[1]\, f(t)\, g(1/t) = 4f(0)$. Thus in order to satisfy $\langle \phi_1(f), \phi_2(g)\rangle = 4[1]\, f(t)\, g(1/t)$, we need to define

$$\varepsilon(f) := 2f(0) - \left\langle \frac{f(t) - f(1/2)}{t - 1/2}, 1 \right\rangle.$$

Obviously under these circumstances, we have $\varepsilon(1) = 2$.

For the remainder of the proof, it suffices to restrict our attention to those g for which $g(0) = g(1) = 0$, so that $\phi_2(g)(t) = (t+1)(t-2)\, g(t)/t(t-1)$. In that case, (3.6) implies

$$\langle t^{-1}, \phi_2(g)\rangle = 2\phi_2(g)(-1) = 0,$$

so for such g, $\langle \phi_1(f), \phi_2(g)\rangle$ is independent of ε.

In particular, if $f(t) = 1$, then we have $\phi_1(f)(t) = \varepsilon(1)\, t^{-1}$, and $\langle \phi_1(f), \phi_2(g)\rangle = 0$. Likewise, we have $[1]\, f(t)\, g(1/t) = [1]\, g(1/t) = g(0) = 0$, as needed.

By linearity, it now suffices to treat the case $f(1/2)=0$. Since $\phi_1(f)(t)=f(t)/(t-1/2)$, we obtain

$$\begin{aligned}
\langle \phi_1(f), \phi_2(g)\rangle &= 2\left(\frac{f(t)}{t-1/2}, \frac{(t+1)(t-2)}{t(t-1)}g(t)\right)\\
&= 2[1]\frac{t}{1+t}\left(\frac{(1/t+1)(1/t-2)f(t)\,g(1/t)}{(t-1/2)(1/t)(1/t-1)}\right.\\
&\qquad \left. -\frac{(t+1)(t-2)f(1/t)\,g(t)}{(1/t-1/2)t(t-1)}\right)\\
&= 4[1]\left(\frac{t}{t-1}f(t)\,g(1/t)+\frac{t}{t-1}f(1/t)\,g(t)\right)\\
&= 4[1]\left(\frac{t}{t-1}f(t)\,g(1/t)+\frac{1/t}{1/t-1}f(t)\,g(1/t)\right)\\
&= 4[1]\,f(t)\,g(1/t). \qquad (4.1)
\end{aligned}$$

Note that all of the expressions that occur in the above calculation are Laurent polynomials (since $g(1)=0$), so we are justified in performing simplifications without addressing questions of convergence. In particular, we can substitute $t\mapsto t^{-1}$ in $tf(1/t)\,g(t)/(t-1)$ (cf. the third equality) without affecting the constant term. ∎

For the analysis of $\mathrm{TS}(2n)$, we need to define another pair of linear transformations, say $\psi_1: \mathbf{Q}[t]^- \to \mathbf{Q}[t]^+$ and $\psi_2: \mathbf{Q}[t]^- \to \mathbf{Q}[t]^-$, by setting

$$\psi_1(f)(t)=f(t)/(t-1/2)$$

$$\psi_2(f)(t)=\frac{(t+1)(t-2)}{t(t-1)}(f(t)+2f(0)(t-1/2))-2f(0)(t-1/2),$$

for all $f\in\mathbf{Q}[t]^-$. As in the previous case, one can easily check that if $u_i^\pm$ $(i=0,1,2,\ldots)$ are monic bases of $\mathbf{Q}[t]^\pm$ with $\deg(u_i^+)=2i$ and $\deg(u_i^-)=2i+1$, then the matrices of ψ_1 and ψ_2 with respect to these bases are upper triangular with unit diagonals. Thus by (3.10), it follows that

$$\mathrm{TS}(2n)=\det[\langle \psi_1(u_i^-), \psi_2(u_j^-)\rangle']_{0\leqslant i,j<n}.$$

Lemma 4.3. *For all $f, g\in\mathbf{Q}[t]^-$, $\langle\psi_1(f), \psi_2(g)\rangle' = 4[1]\,f(t)\,g(1/t)$.*

COROLLARY 4.4. *If* $u_0, u_1, u_2, \ldots \in \mathbf{Q}[t]$ *are monic and* $\deg(u_i) = 2i+1$, *then*

$$\mathrm{TS}(2n) = 2^{2n} \det[c_{ij}]_{0 \leqslant i,j < n},$$

where $c_{ij} = [1]\, u_i(t)\, u_j(t^{-1})$.

Proof of Lemma 4.3. By linearity, it suffices to consider two cases, namely, $g(t) = t - 1/2$ and $g(0) = g(1) = 0$. In the first case, we have $\psi_2(g)(t) = t - 1/2$, so (3.11) and parts (c) and (d) of Lemma 3.1 imply

$$\begin{aligned}\langle \psi_1(f), \psi_2(g)\rangle' &= 2(\psi_1(f), t - 1/2) + 3\psi_1(f)(-1)\\ &= 5\psi_1(f)(0) - 2\psi_1(f)'(0) = 4f'(0) - 2f(0),\end{aligned}$$

whereas $4[1]\, f(t)\, g(1/t) = 4[1](1/t - 1/2)\, f(t) = 4f'(0) - 2f(0)$, as needed.
In the second case, we have $\psi_2(g)(t) = (t+1)(t-2)\, g(t)/t(t-1)$, so $\psi_2(g)(-1) = 0$ and hence

$$\langle \psi_1(f), \psi_2(g)\rangle' = 2\left(\frac{f(t)}{t - 1/2}, \frac{(t+1)(t-2)}{t(t-1)}\, g(t)\right).$$

Although the assumptions on f and g are slightly different, this expression is identical to (4.1). The fact that $\langle \psi_1(f), \psi_2(g)\rangle' = 4[1]\, f(t)\, g(1/t)$ follows by the same reasoning used in the proof of Lemma 4.1. ∎

5. THE CONSEQUENCES

THEOREM 5.1. *For* $n \geqslant 1$, *we have* $2^{n-1}\, \mathrm{CS}(n) = \mathrm{TS}(n)\, \mathrm{TS}(n-1)$.

Proof. Define a symmetric $\mathbf{Q}$-bilinear form on $\mathbf{Q}[t]$ by setting

$$\{f, g\} := [1](f(t)\, g(1/t) + f(1-t)\, g(1 - 1/t))$$

for all $f, g \in \mathbf{Q}[t]$. Note that

$$\{t^i, t^j\} = [1](t^{i-j} + (1-t)^i (1 - 1/t)^j) = \delta_{i,j} + \binom{i+j}{j},$$

so by Lemma 2.4 we have

$$\mathrm{CS}(n) = \det[\{t^i, t^j\}]_{0 \leqslant i,j < n}.$$

More generally, it follows that

$$\mathrm{CS}(n) = \det[\{u_i, u_j\}]_{0 \leqslant i,j < n} \tag{5.1}$$

for any sequence $u_0, u_1, \ldots \in \mathbf{Q}[t]$ with u_i monic and $\deg(u_i) = i$.

On the other hand, note that if $f \in \mathbf{Q}[t]^+$ and $g \in \mathbf{Q}[t]^-$, then we have $f(1-t) = f(t)$ and $g(1-t) = -g(t)$, so $\{f, g\} = 0$ in such cases.

Therefore, let us choose monic bases $u_0^{\pm}, u_1^{\pm}, \ldots$ for $\mathbf{Q}[t]^{\pm}$ with $\deg(u_i^+) = 2i$ and $\deg(u_i^-) = 2i+1$, so that the sequence $u_0^+, u_0^-, u_1^+, u_1^-, \ldots$, satisfies the constraints of (5.1). (One possibility would be to choose $u_i^- = (t-1/2)\,u_i^+ = (t-1/2)^{2i+1}$). With respect to this ordered basis of $\mathbf{Q}[t]$, the matrix of the bilinear form $\{\cdot, \cdot\}$ has zeros in positions (i, j) with $i+j$ odd. Thus after appropriate simultaneous permutations of rows and columns, we obtain a matrix in block-diagonal form, and conclude that

$$\mathrm{CS}(2n) = \det[\{u_i^+, u_j^+\}]_{0 \leq i,j < n} \cdot \det[\{u_i^-, u_j^-\}]_{0 \leq i,j < n}, \tag{5.2a}$$

$$\mathrm{CS}(2n+1) = \det[\{u_i^+, u_j^+\}]_{0 \leq i,j < n} \cdot \det[\{u_i^-, u_j^-\}]_{0 \leq i,j < n}. \tag{5.2b}$$

However, it is easy to see that $\{f, g\} = 2[1]\, f(t)\, g(1/t)$ if $f, g \in \mathbf{Q}[t]^+$ or $f, g \in \mathbf{Q}[t]^-$, so Corollaries 4.2 and 4.4 imply

$$\mathrm{TS}(2n) = 2^n \det[\{u_i^-, u_j^-\}]_{0 \leq i,j < n},$$

$$\mathrm{TS}(2n+1) = 2^n \det[\{u_i^+, u_j^+\}]_{0 \leq i,j \leq n}.$$

Therefore, (5.2a) and (5.2b) yield

$$2^{2n-1}\mathrm{CS}(2n) = \mathrm{TS}(2n-1)\,\mathrm{TS}(2n),$$

$$2^{2n}\mathrm{CS}(2n+1) = \mathrm{TS}(2n+1)\,\mathrm{TS}(2n). \quad ▮$$

For subgroups G of S_3, let $P(G, n) = \prod_{x \in B/G} (r(x)+1)/r(x)$, where $B = B(n, n, n)$ (cf. Theorem 0.2). In the special case $G = S_3$, we obtain

$$\begin{aligned} P(S_3, n) &= \prod_{0 \leq i \leq j \leq k < n} \frac{i+j+k+2}{i+j+k+1} \\ &= \prod_{0 \leq i \leq j < n} \frac{n+i+j+1}{i+2j+1} = \prod_{j=0}^{n-1} \frac{(2j)!\,(n+2j+1)!}{(3j+1)!\,(n+j)!}. \end{aligned} \tag{5.3}$$

COROLLARY 5.2. *For $n \geq 0$, we have*

$$\mathrm{TS}(n) = P(S_3, n) = \prod_{j=0}^{n-1} \frac{(2j)!\,(n+2j+1)!}{(3j+1)!\,(n+j)!}.$$

Proof. By the main theorem of [2] (for alternate proofs, see [5] or [7]), it is known that $\mathrm{CS}(n) = P(C_3, n)$. Hence by induction, it suffices to prove that

$$2^{n-1}P(C_3, n) = P(S_3, n)\, P(S_3, n-1)$$

for $n \geqslant 2$, the basis of the induction being trivial. For this, we note that

$$\{(i, j, k): 0 \leqslant i \leqslant j \leqslant k < n\} \cup \{(k, j, i): 0 \leqslant i < j < k < n\}$$

forms a set of orbit representatives for C_3 on $B(n, n, n)$, so we have

$$\frac{P(C_3, n)}{P(S_3, n)} = \prod_{0 \leqslant i \leqslant j \leqslant k < n} \frac{i+j+k+2}{i+j+k+1} = \prod_{0 \leqslant i \leqslant j < n} \frac{n+i+j+1}{i+2j+2}$$

$$= \prod_{j=0}^{n-2} \frac{(2j+1)!\,(n+2j)!}{(3j+1)!\,(n+j)!},$$

and thus by (5.3), we obtain

$$\frac{P(S_3, n)\,P(S_3, n-1)}{p(C_3, n)} = \prod_{j=0}^{n-2} \frac{n+j}{2j+1} = 2^{n-1}. \quad \blacksquare$$

THEOREM 5.3. *For* $n \geqslant 1$, $\mathrm{TS}(2n-1) = 2^{2n-1}\mathrm{CSTC}(2n)$.

Proof. Define $u_i(t) = t^i(t-1)^i \in \mathbf{Q}[t]^+$ for $i \geqslant 0$. Note that

$$[1]\,u_i(t)\,u_j(t^{-1}) = [1]\,t^{i-j}(t-1)^i(1/t-1)^j = \binom{i+j}{2j-i},$$

so Corollary 4.2 implies

$$\mathrm{TS}(2n-1) = 2^{2n-1}\det\left|\binom{i+j}{2j-i}\right|_{0 \leqslant i,j < n}.$$

Apply Lemma 2.5. $\blacksquare$

Using the formula for $\mathrm{TS}(2n-1)$ in Corollary 5.2, we obtain

COROLLARY 5.4.

$$\mathrm{CSTC}(2n) = \prod_{j=0}^{2n-2} ((2j)!\,(2n+2j)!)/(2(3j+1)!\,(2n+j-1)!).$$

It is routine to verify that this formula agrees with the one originally obtained by Mills, Robbins, and Rumsey in [9], namely

$$\mathrm{CSTC}(2n) = \prod_{i=0}^{n-1} \frac{(3j+1)(6i)!\,(2i)!}{(4i+1)!\,(4i)!}.$$

REFERENCES

1. G. E. ANDREWS, Totally symmetric plane partitions, *Abstracts Amer. Math. Soc.* **1** (1980), 415.
2. G. E. ANDREWS, Plane Partitions. III: The weak Macdonald conjecture, *Invent. Math.* **53** (1979), 193–225.

3. G. E. ANDREWS, Plane Partitions, V: The T.S.S.C.P.P. conjecture, *J. Comb. Theory Ser. A* **66** (1994), 28–39.
4. I. M. GESSEL AND G. VIENNOT, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.
5. G. KUPERBERG, Symmetries of plane partitions and the permanent-determinant method, *J. Comb. Theory Ser. A* **68** (1994), 115–151.
6. P. A. MACMAHON, "Combinatory Analysis," Vol II, Cambridge, 1918; reprinted by Chelsea, New York, 1960.
7. W. H. MILLS, D. P. ROBBINS, AND H. RUMSEY, Proof of the Macdonald conjecture, *Invent. Math.* **66** (1982), 73–87.
8. W. H. MILLS, D. P. ROBBINS, AND H. RUMSEY, Self-complementary totally symmetric plane partitions, *J. Combin. Theory Ser. A* **42** (1986), 277–292.
9. W. H. MILLS, D. P. ROBBINS, AND H. RUMSEY, Enumeration of a symmetry class of plane partitions, *Discrete Math.* **67** (1987), 43–55.
10. R. P. STANLEY, Symmetries of plane partitions, *J. Combin. Theory Ser. A* **43** (1986), 103–113.
11. J. R. STEMBRIDGE, Nonintersecting path, pfaffians, and plane partitions, *Adv. in Math.* **83** (1990), 96–131.